Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 9: The Power Set and the Cartesian Product

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If you discover errors in these solutions or feel you have a better solution, please write to us at udaepp@bucknell.edu or pgorkin@bucknell.edu. We hope that you have fun with these problems.

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Solution to Problem 9.3. *We first calculate* $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$*. Thus,* $\mathcal{P}(\mathcal{P}(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$.

Solution to Problem 9.6. *(a) We claim that if* $X \neq \emptyset$ *and* $X \in \mathcal{P}(A \setminus B)$*, then* $X \in \mathcal{P}(A) \setminus \mathcal{P}(B)$*.*

If $X \neq \emptyset$ *and* $X \in \mathcal{P}(A \setminus B)$*, then* $X \subseteq A \setminus B$*. Thus* $X \subseteq A$ *and hence* $X \in \mathcal{P}(A)$*. Suppose that* $X \in \mathcal{P}(B)$*. Then* $X \subseteq B$*, which would imply that* $X \nsubseteq A \setminus B$*. This contradicts the earlier finding and hence* $X \notin \mathcal{P}(B)$*. We conclude that* $X \in \mathcal{P}(A) \setminus \mathcal{P}(B)$*, establishing the claim.*

(b) For any set Z *we have* $\emptyset \in \mathcal{P}(Z)$ *by Theorem 6.10. Hence* $\emptyset \in \mathcal{P}(A \setminus B)$*, but* $\emptyset \notin \mathcal{P}(A) \setminus \mathcal{P}(B)$*. This shows that* $\mathcal{P}(A \setminus B) \neq \mathcal{P}(A) \setminus \mathcal{P}(B)$*.*

Solution to Problem 9.9. *If* $X \in \bigcup_{\alpha \in I} \mathcal{P}(A_\alpha)$*, then* $X \in \mathcal{P}(A_\alpha)$ *for some* $\alpha \in I$*. Hence* $X \subseteq A_\alpha$ *for some* $\alpha \in I$*. We claim that* $X \subseteq \bigcup_{\alpha \in I} A_\alpha$*. If* $y \in X$*, then* $y \in A_\alpha$ *for some* $\alpha \in I$*. Hence* $y \in \bigcup_{\alpha \in I} A_\alpha$*. This establishes the claim. Since* $X \subseteq \bigcup_{\alpha \in I} A_\alpha$*, we conclude that* $X \in \mathcal{P}(\bigcup_{\alpha \in I} A_\alpha)$*. This establishes the set inclusion.*

Solution to Problem 9.12. *(a) This set contains of all the points in the plane that have integer coordinates, are on or to the right of the y-axis, and are on or below the diagonal line* $y = x$*.*

(b) This set contains all points in the plane that are on the diagonal line $y = x$*.*

(c) This set contains all points in the plane that have integer coordinates and both coordinates of each point are even or both coordinates are odd.

(d) This set contains all the points with x-coordinate 0 or 1 and with the y-coordinate any non-negative integer.

(e) *This set is the graph of the function* $y = x^2$ *in the* xy*-coordinate plane.*

(f) *This set contains all points on the graph of* $y = x^2$ *that have positive* x*-coordinate and the* y*-coordinate is an integer.*

Solution to Problem 9.15. *If* $(x, y) \in \mathbb{N} \times \mathbb{N}$*, then* $x \in \mathbb{N}$ *and* $y \in \mathbb{N}$*. Since* $\mathbb{N} \subseteq \mathbb{Z}$*, we conclude that* $x \in \mathbb{Z}$ *and* $y \in \mathbb{Z}$*. Hence* $(x, y) \in \mathbb{Z} \times \mathbb{Z}$*. We have established that* $\mathbb{N} \times \mathbb{N} \subseteq \mathbb{Z} \times \mathbb{Z}$*.*

Solution to Problem 9.18. *The answer is no. Consider* $A = \mathbb{R}$*,* $B = \emptyset$*,* $C = \mathbb{Z}$*, and* $D = \mathbb{N}$*. Then* $\mathbb{R} \times \emptyset = \emptyset \subseteq \mathbb{Z} \times \mathbb{N}$*. Of course* $\mathbb{R} \not\subseteq \mathbb{Z}$*.*

Note that if we add the condition that $B \neq \emptyset$*, then* $A \subseteq C$ *follows. If instead we add the condition that* $A \neq \emptyset$*, then* $B \subseteq D$*. If we insist on both,* $A \neq \emptyset$ *and* $B \neq \emptyset$*, then* $A \times B \subseteq C \times D$ *implies that* $A \subseteq C$ *and* $B \subseteq D$*. Make sure you understand how these additional conditions allow you to prove the conclusion!*

Solution to Problem 9.21. *(a)* $R = \{(0, \{0\}), (0, \{0, 1\}), (1, \{1\}), (1, \{0, 1\})\}$

(b) *This is the "element of" relation.*