Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 8: More on Operations on Sets

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Solution to Problem 8.3. *We claim that $A = \{x \in \mathbb{R} : x \geq 0\}$.*

Proof. If $y \in A = \bigcup_{j=0}^{\infty}[j, j+1]$, then $y \in [j, j+1]$ for some $j \in \mathbb{N}$. Hence $y \in \mathbb{R}$ and $0 \leq j \leq y \leq j+1$. This implies that $y \in \{x \in \mathbb{R} : x \geq 0\}$. Hence $A \subseteq \{x \in \mathbb{R} : x \geq 0\}$.

Conversely, if $y \in \{x \in \mathbb{R} : x \geq 0\}$, then we let $k = \lfloor y \rfloor$ which is defined to be the greatest integer less or equal to y. Since $y \geq 0$ we conclude that $k \geq 0$ and the definition of the greatest integer implies that $k \leq y < k+1$. Hence $y \in [k, k+1]$ for some $k \in \mathbb{N}$. We conclude that $y \in \bigcup_{j=0}^{\infty}[j, j+1] = A$. Thus $\{x \in \mathbb{R} : x \geq 0\} \subseteq A$.

The two parts together show that $A = \{x \in \mathbb{R} : x \geq 0\}$. □

We claim that $B = \mathbb{Z}$.

Proof. If $y \in B = \bigcap_{j \in \mathbb{Z}}(\mathbb{R} \setminus (j, j+1))$, then $y \in \mathbb{R} \setminus (j, j+1)$ for all $j \in \mathbb{Z}$. Then $y \in \mathbb{R}$ and $y \notin (j, j+1)$ for all $j \in \mathbb{Z}$. Hence $y = k$ for some $k \in \mathbb{Z}$. That is, $y \in \mathbb{Z}$. We conclude that $B \subseteq \mathbb{Z}$.

Conversely, if $y \in \mathbb{Z}$, then $y \in \mathbb{R}$ and $y \notin (k, k+1)$ for all $k \in \mathbb{Z}$. Hence $y \in \mathbb{R} \setminus (k, k+1)$ for all $k \in \mathbb{Z}$. Therefore, $y \in \bigcap_{j \in \mathbb{Z}}(\mathbb{R} \setminus (j, j+1)) = B$. This means that $\mathbb{Z} \subseteq B$.

The two parts together imply that $B = \mathbb{Z}$. □

Solution to Problem 8.6. *This statement is false, we give a counterexample.*

For $n \in \mathbb{Z}^+$ we define $A_n = [0, 1/n)$ and $B_n = [0, 1/n]$, the half open and closed interval of reals, respectively. It is clear that $A_n \subset B_n$ for all $n \in \mathbb{Z}^+$. It is also easily seen that

$$\bigcap_{n \in \mathbb{Z}^+} A_n = \bigcap_{n \in \mathbb{Z}^+} B_n = \{0\}.$$

Solution to Problem 8.9. *If $x \in \bigcup_{n\in\mathbb{Z}^+} A_n$, then $x \in A_{n_0}$ for some $n_0 \in \mathbb{Z}^+$; that is,*

$$0 < \frac{1}{n_0} < x \leq 2 < \frac{3}{n} + 2 \quad \text{for all } n \in \mathbb{Z}^+.$$

Hence $x \in B_n$ for all $n \in \mathbb{Z}^+$. Thus, $x \in \bigcap_{n\in\mathbb{Z}^+} B_n$ and $\bigcup_{n\in\mathbb{Z}^+} A_n \subseteq \bigcap_{n\in\mathbb{Z}^+} B_n$.

Solution to Problem 8.12. *(a) If $x \in \left(\bigcup_{\alpha\in I} A_\alpha\right) \cap B$, then $x \in \bigcup_{\alpha\in I} A_\alpha$ and $x \in B$. Thus $x \in A_\alpha$ for some $\alpha \in I$ and $x \in B$. This implies that $x \in A_\alpha \cap B$ for some $\alpha \in I$. Hence $x \in \bigcup_{\alpha\in I} (A_\alpha \cap B)$. Consequently, $\left(\bigcup_{\alpha\in I} A_\alpha\right) \cap B \subseteq \bigcup_{\alpha\in I} (A_\alpha \cap B)$.*

Conversely, if $x \in \bigcup_{\alpha\in I} (A_\alpha \cap B)$, then $x \in A_\alpha \cap B$ for some $\alpha \in I$. Hence $x \in A_\alpha$ for some $\alpha \in I$ and $x \in B$. Thus, $x \in \bigcup_{\alpha\in I} A_\alpha$ and $x \in B$. That is, $x \in \left(\bigcup_{\alpha\in I} A_\alpha\right) \cap B$. This shows that $\bigcup_{\alpha\in I} (A_\alpha \cap B) \subseteq \left(\bigcup_{\alpha\in I} A_\alpha\right) \cap B$.

The two parts together show the equality of the two sets.

(b) We claim that under the given conditions the following set equality holds:

$$\left(\bigcap_{\alpha\in I} A_\alpha\right) \cup B = \bigcap_{\alpha\in I} (A_\alpha \cup B).$$

If $x \in \left(\bigcap_{\alpha\in I} A_\alpha\right) \cup B$, then $x \in \bigcap_{\alpha\in I} A_\alpha$ or $x \in B$. Hence $x \in A_\alpha$ for all $\alpha \in I$ or $x \in B$. This implies that $x \in A_\alpha \cup B$ for all $\alpha \in I$. (Why?) Thus, $x \in \bigcap_{\alpha\in I} (A_\alpha \cup B)$. This shows that $\left(\bigcap_{\alpha\in I} A_\alpha\right) \cup B \subseteq \bigcap_{\alpha\in I} (A_\alpha \cup B)$.

Conversely, if $x \in \bigcap_{\alpha\in I} (A_\alpha \cup B)$, then $x \in A_\alpha \cup B$ for all $\alpha \in I$. Thus $x \in B$ or $x \in A_\alpha$ for all $\alpha \in I$. Hence $x \in B$ or $x \in \bigcap_{\alpha\in I} A_\alpha$. (Why?) This implies that $x \in \left(\bigcap_{\alpha\in I} A_\alpha\right) \cup B$. Consequently, $\bigcap_{\alpha\in I} (A_\alpha \cup B) \subseteq \left(\bigcap_{\alpha\in I} A_\alpha\right) \cup B$.

The two parts together establish the claim.

Solution to Problem 8.15. *Claim: $A = 2\mathbb{Z}$.*

Proof. If $x \in 2\mathbb{Z}$, then $x \in \mathbb{Q}$. Also, $x = 2m$ for some $m \in \mathbb{Z}$. Thus $x \notin \mathbb{R} \setminus \{2m\}$ for some $m \in \mathbb{Z}$. Hence $x \notin \bigcap_{n\in\mathbb{Z}} (\mathbb{R} \setminus \{2n\})$. Hence $x \in \mathbb{Q} \setminus \bigcap_{n\in\mathbb{Z}} (\mathbb{R} \setminus \{2n\}) = A$. thus $2\mathbb{Z} \subseteq A$.

Conversely, if $x \in A$, then $x \in \mathbb{Q} \setminus \bigcap_{n\in\mathbb{Z}} (\mathbb{R} \setminus \{2n\})$. That is, $x \in \mathbb{Q}$ and $x \notin \bigcap_{n\in\mathbb{Z}} (\mathbb{R} \setminus \{2n\})$. Thus $x \notin \mathbb{R} \setminus \{2n\}$ for some $n \in \mathbb{Z}$. Since $x \in \mathbb{Q} \subseteq \mathbb{R}$, we conclude that that $x \in \{2n\}$ for some $n \in \mathbb{Z}$. That is, $x = 2n$ for some $n \in \mathbb{Z}$. Hence $x \in 2\mathbb{Z}$. This shows that $A \subseteq \mathbb{Z}$.

This finishes the proof that $A = 2\mathbb{Z}$.

$\square$

Solution to Problem 8.18. (a) *Let* $A_n = \{x \in \mathbb{R} : n \leq x < n+1\}$. *Then the collection* $\mathcal{A} = \{A_n : n \in \mathbb{Z}^+\}$ *is pairwise disjoint: If* $A_n, A_m \in \mathcal{A}$ *with* $A_n \cap A_m \neq \emptyset$, *then there is* $x \in A_n \cap A_m$. *This implies that* $n \leq x < m+1$. *Hence* $n - m < 1$. *Since both* m *and* n *are positive integers, we conclude that* $n = m$. *Hence* $A_n = A_m$. *This shows that* $\mathcal{A}$ *is pairwise disjoint.*

(b) *The contrapositive is: "If* $X \neq Y$, *then* $X \cap Y = \emptyset$."

(c) *The converse is: "If* $X = Y$, *then* $X \cap Y \neq \emptyset$."

(d) *Yes, it does hold. The reason is that the assertion of (b) is the contrapositive of the defining condition of pairwise disjoint collection. The contrapositive is logically equivalent with the original statement.*

(e) *Yes the set* $\mathcal{A}$ *is a pairwise disjoint collection. The statement of part (b) is equivalent to the defining statement of pairwise disjoint collection.*

(f) *In the trivial case,* $\mathcal{B} = \{B\}$, *where* $B \neq \emptyset$, *this is false. In all other cases, this is true. We begin with the trivial case. In that case,* $\mathcal{B} = \{B\}$, *where* $B \neq \emptyset$, *thus* $\mathcal{B}$ *is pairwise disjoint. But we have* $\bigcap_{X \in \mathcal{B}} X = B \neq \emptyset$. *(Recall that we have to assume that* $\mathcal{B} \neq \emptyset$ *because the intersection of an empty collection of sets is not defined!)*

However, if $\mathcal{B}$ *has at least two elements, then* $\bigcap_{X \in \mathcal{B}} X = \emptyset$: *Suppose not, then there is* $x \in \bigcap_{X \in \mathcal{B}} X$. *Let* X_1 *and* X_2 *be two elements of* $\mathcal{B}$. *(We may assume that they are different because a set that has two elements, both the same, is not considered to have two elements.) Then* $x \in X_1$ *and* $x \in X_2$. *This contradicts the condition of being pairwise disjoint.*

(g) *No, it need not be pairwise disjoint. Consider* $\mathcal{B} = \{[0,3], [2,5], [4,7]\}$. *Here the sets denote closed intervals of the reals. Then* $[0,3] \cap [2,5] = [2,3] \neq \emptyset$ *and thus* $\mathcal{B}$ *is not pairwise disjoint. But* $\bigcap_{X \in \mathcal{B}} = [0,3] \cap [2,5] \cap [4,7] = \emptyset$.

Solution to Problem 8.21. *There are many examples. For example, let* $A_j = [-1/j, \infty) = \{x \in \mathbb{R} : x \geq -1/j\}$. *We clearly have* $A_{j+1} \subset A_j$ *for all* $j \in \mathbb{Z}^+$. *Also,* $\bigcap_{j=1}^{\infty} A_j = \{x \in \mathbb{R} : x \geq 0\} \neq \emptyset$.