Reading, Writing, and Proving (Second Edition)

Ulrich Daepp and Pamela Gorkin

Springer Verlag, 2011

Solutions to Chapter 6: Sets

A Note to Student Users. Check with your instructor before using these solutions. If you are expected to work without any help, do not use them. If your instructor allows you to find help here, then we give you permission to use our solutions provided you credit us properly.

If you discover errors in these solutions or feel you have a better solution, please write to us at udaepp@bucknell.edu or pgorkin@bucknell.edu. We hope that you have fun with these problems.

Ueli Daepp and Pam Gorkin

Solution to Problem 6.3. *Solutions appear below.*

(a) $x \in \mathbb{R}$;

(b) $\mathbb{Z} \subset \mathbb{R}$ *(some authors view this as proper subset, others do not; if your teacher uses different notation, you will have to write* $\mathbb{Z} \subseteq \mathbb{R}$ *and* $\mathbb{Z} \neq \mathbb{R}$*).*

(c) $x \in \mathbb{Z} \rightarrow \big((x \in \mathbb{N}) \vee (-x \in \mathbb{N})\big)$;

(d) $\{(3n)^2 : n \in \mathbb{Z}\}$.

Solution to Problem 6.6. *(a) The figure on the left is* $X \setminus (A \cup B)$ *or* $(A \cup B)^c$;

(b) The figure on the right is $(A \cup B)^c \cup (A \cap B)$.

Solution to Problem 6.9. *(a) The upper-left figure is* $B \setminus A$;

(b) The upper-right figure is $(A \cup B) \setminus (A \cap B)$;

(c) The figure on the left in the middle is $A \cap B \cap C$;

(d) The figure on the right in the middle is $(B \cap C) \setminus (A \cap B \cap C)$;

(e) Each of the shaded regions in the bottom figure can be written, with appropriate changes to the names of the sets, as we did in the previous part of the problem. This is the union of those three. Alternatively, this is $((A \cap C) \cup (A \cap B) \cup (B \cap C)) \setminus (A \cap B \cap C)$.

Solution to Problem 6.12. *Let $x \in A \cap B$. Then $x \in A$, so 6 divides x, and there is an integer m such that $x = 6m$. Further, $x \in B$ so 21 divides x and there is an integer k such that $x = 21k$. Therefore, $2, 3$, and 7 are factors of x. Consequently, 42 divides x and we see that $x \in C$.*
For the other direction, if $x \in C$, then 42 divides x. Therefore $x = 42l$ for some integer l. So $x = 6(7l) = 21(2l)$ and $x \in A \cap B$.

Solution to Problem 6.15. *No. Let $x = y = -1/2$. Then $x, y \in S$ but $x \sharp y = 0 \notin S$.*

Solution to Problem 6.18. *(a) This is set of all points in the Cartesian plane with the x-axis removed.*

(b) If we begin with (x, y) and (z, w) in our set, then $y \neq 0$ and $w \neq 0$. Therefore $wy \neq 0$, and $(x, y) \diamond (z, w) = (xw + zy, wy) \in A$.

(c) Consider the element $(0, 1)$. Then $(0, 1) \in A$ and

$$(0, 1) \diamond (x, y) = (x, y),$$

for every $(x, y) \in A$.

(d) Yes! It's very familiar. Think of (x, y) as the representation of a fraction x/y. Then this new addition is precisely the way we add fractions.

Solution to Problem 6.21. *Suppose that there exists an element $x \in A$. Since $x \in \mathbb{N}$ and $x < x^2$, we conclude that $x \geq 2$. Choosing $y = x \in \mathbb{Z}$, we have $y|x$. We conclude that $y^2|x$ and thus $x^2|x$; that is $x = kx^2$ for some $k \in \mathbb{Z}$. So $1 = kx$ with $x, k \in \mathbb{N}$ and $x \geq 2$. This is a contradiction and shows that A is empty.*

Solution to Problem 6.24.

If $(x, y) \in A \cap B$, then $(x, y) \in A$ and therefore x and y are either both positive or both negative. Suppose that they are both negative. Then $(x, y) \in B$ implies that $y > -x$. But that's silly, because $-x$ is positive and y is negative. Therefore, this case does not arise. So they are both positive. Then $(x, y) \in B$ implies that $y > x > 0$ and we see that $(x, y) \in C$. So, $A \cap B \subseteq C$.
Now suppose $(r, s) \in C$. Then $0 < r < s$. Thus, $rs > 0$ and $rs \in A$. Further, $s > r = |r|$, so $(r, s) \in B$. Consequently, $(r, s) \in A \cap B$. So, $C \subseteq A \cap B$.