Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 5: Proof Techniques

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Solution to Problem 5.3. *Let n be an integer. Then $n^2 + 3n + 2 = (n + 1)(n + 2)$. Now $n + 1$ and $n + 2$ are consecutive integers, so one of them must be divisible by 2; that is, one of them must equal $2m$ for some integer m. If $n + 1 = 2m$, then $n^2 + 3n + 2 = 2m(n + 2)$. Since $m(n + 2)$ is an integer, we see that $n^2 + 3n + 2$ is even. If $n + 2$ is divisible by 2, the same argument establishes the desired result.*

Solution to Problem 5.6. *(a) For Statement 1, it seems prudent to prove the contrapositive statement directly. This seems appropriate because the statement is essentially of the form $\neg A \to \neg B$. The contrapositive turns this statement into a "positive" problem.*

The same method could work for Statement 2 for essentially the same reason. The negation of the conclusion of the statement is simpler than the original conclusion. However, the conclusion of the contrapositive is relatively complicated: "x is irrational or y is rational." The antecedent of the original statement is simpler: "x is rational and y is irrational." Using the method of contradiction will essentially use simpler forms at both ends and thus will likely lead to the most efficient solution.

(b) We will prove the contrapositive and so we will start by assuming that x and y are both rational. Hence, there are integers $p, q, r,$ and s with $q \neq 0$ and $r \neq 0$ such that $x = p/q$ and $y = r/s$. Then $xy = (pr)/(qs)$ where pr and qs are both integers and $qs \neq 0$. Thus xy is a rational number and we have proven the contrapositive of Statement 1. So Statement 1 also holds.

The proof went through as expected.

(c) We will prove Statement 2 by contradiction. So suppose to the contrary, that x is rational, y is irrational and $x + y$ is also rational. Then there are integers $p, q, r,$ and s with $q \neq 0$ and $r \neq 0$ such that $x = p/q$ and $x + y = r/s$. We solve for y to get $y = r/s - p/q = (rq - ps)/(qs)$. Since $rq - ps$ and qs are both integers and $qs \neq 0$ we conclude that y is rational, giving us the desired contradiction. We have established the truth of Statement 2.

The proof went through as expected.

Solution to Problem 5.9. *We wish to prove that if n^2 is divisible by 3, then n is divisible by 3. It might be easier to prove the contrapositive, so suppose that n is not divisible by 3. Then we have $n = 3m + 1$ or $n = 3m + 2$ for some integer m. (Why?) We consider these two cases separately.*
Case 1. If $n = 3m + 1$, then $n^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1$. Therefore, in this case, n^2 is not divisible by 3. (Why?)
Case 2. If $n = 3m + 2$, then $n^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1$. Therefore, in this case, n^2 is not divisible by 3. (Why?)
We have shown that if n is not divisible by 3, the n^2 is not divisible by 3. This establishes the result.

Solution to Problem 5.12. *Let n be an integer. With respect to division by 3, we have three cases for n.*
Case 1. n is divisible by 3. Then $n = 3m$ for some integer m and $n^2 = 3(3m^2)$ with $3m^2$ an integer. Thus, n is not of the form $3k + 2$.
Case 2. n has a remainder of 1 after division by 3. Then $n = 3m + 1$ for some integer m and $n^2 = 3(3m^2 + 2m) + 1$ with $3m^2 + 2m$ an integer. Again, n is not of the form $3k + 2$.
Case 3. n has a remainder of 2 after division by 3. Then $n = 3m + 2$ for some integer m and $n^2 = 3(3m^2 + 4m + 1) + 1$ with $3m^2 + 4m + 1$ an integer. Again we see that n is not of the form $3k + 2$.
Since this exhausts all possibilities, the square of an integer cannot be of the form $3k + 2$ for an integer k.

Solution to Problem 5.15. *This can be done in cases, but that would be painful. Instead, we use the triangle inequality appearing in the previous problem. Thus, we add zero creatively to obtain*

$$|x| = |x - y + y| \leq |x - y| + |y| \text{ or } |x| - |y| \leq |x - y|.$$

Interchanging the roles of x and y, we obtain $|y| - |x| \leq |x - y|$. Putting these together, we obtain

$$-|x - y| \leq |x| - |y| \leq |x - y|.$$

This is precisely what we mean by $||x| - |y|| \leq |x - y|$.

Solution to Problem 5.18. *This conjecture is false. We have a counterexample: $x = 19$ and $y = 9$. (Check it.)*

There are other examples: Another counterexample would have been $x = 8$ and $y = 3$. How did we find them?

Solution to Problem 5.21. *This looks like a job for the contrapositive! So suppose that n is odd. Then there is an integer m such that $n = 2m + 1$. Thus*

$$n^2 - (n - 2)^2 = 8m.$$

This is obviously divisible by 8. Therefore, we have shown that if n is odd, then $n^2 - (n - 2)^2$ is divisible by 8. Since this is the contrapositive of what we needed to show, we have established the result.

Solution to Problem 5.24. *(a) We must prove that for all positive integers x, there exist nonzero rational numbers y and z such that $x^2 = y^2 + z^2$.*

(b) Note that since x is an integer and $x \neq 0$, this is the same as showing that

$$1 = (y/x)^2 + (z/x)^2.$$

So let $y = 3x/5$ be one rational number and $z = 4x/5$ be the second. Then

$$(y/x)^2 + (z/x)^2 = (3/5)^2 + (4/5)^2 = 1,$$

as desired.

Solution to Problem 5.27. *Note: This problem should have said "give an example,* **if possible***, of a simply even s-difference."*

(a) There are no examples! (See below for a proof.)

Proof. Suppose to the contrary that there is a simple s-difference. That is, there is a natural number x and natural numbers y and z such that $x = y^2 - z^2$. Then $x = (y - z)(y + z)$. Since x is even, $y - z$ or $y + z$ must be even. For $y - z$ to be even we need y and z to both be even or both odd. In both cases, $y + z$ is also even and thus x is divisible by 4. The same is true if we require $y + z$ to be even. Hence there are no simply even s-differences. □

Note that therefore any even natural number that is not divisible by 4 is an example of a natural number that is not an s-difference. For instance, $2, 6, 10$. This could be used to answer Problem 26 (c).

(b) 0, 456789 *(any natural number is a nonexample).*

(c) Of course not. A definition that defines the empty set is not relevant. (Note though, that other than the fact that it is an empty definition, it is not strictly an incorrect definition!)