Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 4: Set Notation and Quantifiers

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Solution to Problem 4.3. *(a) This asks whether for every real number x, there is an integer y such that $x = 2y$. This is false; for example, we may take $x = 5$. Then $x \neq 2y$ for any integer y.*

(b) Since every even integer is a real number, this is true.

(c) The example from the first part of this problem shows that this is false. Other examples are possible.

(d) This is true. For some real number x (namely, an even integer $x = 2n$) there exists an integer y (namely, $y = n$) such that $x = 2y$.

(e) The answer to this is the same as the previous part of this problem.

Solution to Problem 4.6. *We work the first part in detail. The second part will follow in much the same way.*

(a) This says "For every x we have $x \in A$ if and only if there exists an n such that n is an integer and $x = 2n$." Thus, x is an element of A if and only if x is an even number. In other words, $A = 2\mathbb{Z}$.

(b) Similarly, $B = 2\mathbb{Z} + 1$, or the odd numbers.

Solution to Problem 4.9. *(a) The negation is*

$$\exists x, ((x \in \mathbb{Z} \wedge \neg(\exists y, (y \in \mathbb{Z} \wedge x = 7y)) \wedge (\forall z, (z \notin \mathbb{Z} \vee x \neq 2z)).$$

(b) For every x, if x is an integer and there exists no integer y such that y such that $x = 7y$, then there exists an integer z such that $x = 2z$.

(c) The original statement is false, so the negation is true. As an example, if $x = 3$ then there is no integer y such that $3 = 7y$ so the antecedent is true, but there is no integer z such that $3 = 2z$ and the conclusion is false.

Solution to Problem 4.12. *(a) The standard form of this implication is: "If Madeleine waters the plants, then it is Tuesday."*

1. *"Madeleine waters the rosebush and it is not Tuesday."*
2. *"If it is Tuesday, then Madeleine waters the rosebush."*
3. *"It is Tuesday and Madeleine does not water the rosebush."*
4. *"If it is not Tuesday, then Madeleine does not water the rosebush."*
5. *Same as in 1.*

(b)
1. *"I ski and I will not fall."*
2. *"If I will fall, then I ski."*
3. *"I will fall and I do not ski."*
4. *"If I will not fall, then I do not ski."*
5. *Same as in 1.*

(c) We talk about all windows and balls, so we will use the variable w with the universe of all windows and the variable b with the universe of all balls. The original statement can then be rewritten as: "$\forall w, \forall b,$ (You throw b through $w \rightarrow w$ breaks)."

1. *"$\exists w, \exists b,$ (you throw b through w and w does not break). In English: "You throw some ball through some window and that window does not break."*
2. *$\forall w, \forall b,$ (w breaks $\rightarrow$ you throw b through w). In English: "If windows break then you throw balls through them."*
3. *$\exists w, \exists b,$ (w breaks $\wedge$ you do not throw b through w). In English: "Some window breaks and there is a ball that you do not throw through them."*
4. *$\forall w, \forall b,$ (w does not break $\rightarrow$ you do not throw b through w). In English: "If windows do not break then you do not throw balls through them."*
5. *Same as in 1.*

(d) We use the variable s with all sentences as the universe. The statement is then: $\forall s,$ (I negate $s \rightarrow I$ write s incorrectly).

1. *$\exists s,$ (I negate $s \wedge I$ write s correctly). In English: "There is a sentence that I negate correctly."*
2. *$\forall s,$ (I write s incorrectly $\rightarrow I$ negate s). In English: "If I make a mistake in a sentence, then I negated it."*
3. *$\exists s,$ (I write s incorrectly $\wedge I$ do not negate s). In English: "There is a sentence that I do not negate but write incorrectly."*
4. *$\forall s,$ (I write s correctly $\rightarrow I$ do not negate s). In English: "If I have a sentence correct then I did not negate it."*
5. *Same as in 1.*

(e) Standard form: "If I come, then you invite me."

1. *"I come and you do not invite me."*
2. *"If you invite me, then I come."*
3. *"You invite me and I do not come."*
4. *"If you do not invite me, then I do not come."*
5. *Same as in 1.*

(f) *We will use* $\mathbb{R}$ *for the universe of* x *and* $\mathbb{Z}$ *for the universe of* n*. The statement is then:*
$\forall x, (x > 0 \rightarrow \exists n, 1/n < x)$.

1. $\exists x, (x > 0 \wedge \forall n, 1/n \geq x)$. *"There is a positive real number* x *such that* $x \leq 1/n$ *for all integers* n*."*
2. $\forall x, (\exists n, (1/n < x) \rightarrow (x > 0))$. *" For every real number* x*, if there is an integer* n *such that* $1/n < x$*, then* x *is positive."*
3. $\exists x, (\exists n, (1/n < x) \wedge (x \leq 0))$. *"There is a nonpositive real number* x *such that* $1/n < x$ *for some integer* n*."*
4. $\forall x, (\forall n, (1/n \geq x) \rightarrow (x \leq 0))$. *"If for real numbers* x *we have* $1/n \geq x$ *for all integers* n*, then* x *is nonpositive."*
5. *Same as in 1.*

(g) *Here the universe is* $\mathbb{R}$ *and the statement is:* $\forall x, (x \neq 0 \rightarrow x^2 \neq 0)$.

1. $\exists x, (x \neq 0 \wedge x^2 = 0)$.
2. $\forall x, (x^2 \neq 0 \rightarrow x \neq 0)$.
3. $\exists x, (x^2 \neq 0 \wedge x = 0)$.
4. $\forall x, (x^2 = 0 \rightarrow x = 0)$.
5. *Same as in 1.*

(h) *The universe for* x *and* y *is* $\mathbb{R}$*.* $\forall x, (x \neq 0 \rightarrow \exists y, xy = 1)$.

1. $\exists x, (x \neq 0 \wedge \forall y, xy \neq 1)$.
2. $\forall x, (\exists y, xy = 1 \rightarrow x \neq 0)$.
3. $\exists x, (\exists y, xy = 1 \wedge x = 0)$.
4. $\forall x, (\forall y, xy \neq 1 \rightarrow x = 0)$.
5. *Same as in 1.*

(i) *We will use the variables* $x, y, m, n,$ *and* k*. All of them will have* $\mathbb{Z}$ *as the universe. The original statement is:* $\forall x, \forall y, ((\exists m, x = 2m \wedge \exists n, y = 2n) \rightarrow \exists k, x + y = 2k)$.

1. $\exists x, \exists y, (\exists m, x = 2m \wedge \exists n, y = 2n \wedge \exists k, x + y = 2k + 1)$. *"There are even integers* x *and* y *such that* $x + y$ *is an odd integer."*
2. $\forall x, \forall y, (\exists m, x + y = 2m \rightarrow (\exists n, x = 2n \wedge \exists k, y = 2k))$. *"If the sum of two integers is even, then each integer itself is even."*

3. $\exists x, \exists y, (\exists m, x + y = 2m \wedge (\exists n, x = 2n + 1 \vee \exists k, y = 2k + 1))$. *"There are integers, x and y, at least one of them odd, such that $x + y$ is even."*

4. $\forall x, \forall y, (\exists m, x + y = 2m + 1 \rightarrow (\exists n, x = 2n + 1 \vee \exists k, y = 2k + 1))$. *"If the sum of two integers is odd, then at least one of the integers is odd."*

5. *Same as in 1.*

Solution to Problem 4.15. *Note that we assume the universe is $\mathbb{R}$.*

(a) *For all M, we have $M \notin \mathbb{Z}$ or there exists an x such that $x^2 > M$.*

(b) *Given any real number M, there exists a real number x such that $x^2 > M$ (for example, we may take $x = M + 1$). Therefore the negation of the statement is true.*

Solution to Problem 4.18. *(a) We suggest taking the universe to be $\mathbb{Z}$. We will introduce another variable n that will use the same universe. (There are other possibilities, e.g. $\mathbb{R}$. But choosing the integers will make the problem simpler.)*

(b) $\forall x, (8 \nmid (x^2 - 1) \rightarrow \exists n, (x = 2n))$.

(c) $\exists x, (8 \nmid (x^2 - 1) \wedge \exists n, (x = 2n + 1))$. *In words: "There is an odd integer x such that 8 does not divide $x^2 - 1$."*