Reading, Writing, and Proving (Second Edition)

Ulrich Daepp and Pamela Gorkin

Springer Verlag, 2011

Solutions to Chapter 24: The Cantor-Schröder-Bernstein Theorem

A Note to Student Users. Check with your instructor before using these solutions. If you are expected to work without any help, do not use them. If your instructor allows you to find help here, then we give you permission to use our solutions provided you credit us properly.

If you discover errors in these solutions or feel you have a better solution, please write to us at udaepp@bucknell.edu or pgorkin@bucknell.edu. We hope that you have fun with these problems.

Ueli Daepp and Pam Gorkin

Solution to Problem 24.3. *Claim 1. If $B_b \neq \emptyset$, then $g\,|_{B_b} : B_b \longrightarrow A_b$ is a bijection.*

Let $y \in B$. Then $x = g(y) \in A$ with corresponding sequence $(x_n) = (x,\ y,\ x_2,\ \ldots)$ while the sequence of y is $(y,\ y_1,\ \ldots)$ with $y_k = x_{k+1}$ for $k \geq 0$. Hence, if $y \in B_b$, then the sequence (y_n) is of odd length and thus the sequence (x_n) is of even length, so $x \in A_b$. This implies that range$(g\,|_{B_b}) \subseteq A_b$. If $x \in A_b$, then the sequence (x_n) is of even length and has at least two terms $(x,\ x_1,\ \ldots)$. We set $y = x_1$ and note that $x = g(y)$. As above the sequence of y satisfies $y_k = x_{k+1}$ for $k \geq 0$. Since (x_n) is of even length, (y_n) is of odd length and thus $y \in B_b$. This shows that $A_b \subseteq$ range$(g\,|_{B_b})$.
We have now shown that $g\,|_{B_b} : B_b \longrightarrow A_b$ is well-defined and is surjective. Since $g\,|_{B_b}$ is the restriction of an injection, it is also one-to-one. This establishes the first claim.

Claim 2. If $B_b = \emptyset$ then $A_b = \emptyset$.

If $A_b \neq \emptyset$, then there is $a \in A$ such that the sequence (x_n) is of even length of at least two. Setting $y = x_1 \in B$, the sequence (y_n) satisfies $y_k = x_{k+1}$ for all $k \geq 0$. In particular, (y_n) is of odd length and thus $B_b \neq \emptyset$. This establishes the second claim.

The two claims together show that $A_b \approx B_b$.

Solution to Problem 24.6. *If $m = 0$, then $X = \emptyset$ and the unique function $f : \emptyset \longrightarrow Y$ is trivially injective. Thus $|X| \leq |Y|$.*

Assume now that $m > 0$. Using the definition for finite cardinality we have maps

$$X \xrightarrow{f} \{1, \ldots, m\} \xrightarrow{g} \{1, \ldots, n\} \xrightarrow{h} Y$$

where f and h are bijections and $g(x) = x$ is an injection. Thus $h \circ g \circ f : X \longrightarrow Y$ is an injection, showing that $|X| \leq |Y|$.

Solution to Problem 24.9. *(a) We define a function f by*

$$f(x) = 2^a 3^p 5^q, \text{ where } a = \begin{cases} 0 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases} \quad \text{and } \frac{p}{q} = |x| \text{ with } gcd(p,q) = 1,$$

where p and q are integers, $p \geq 0$ and $q > 0$. You have to convince yourself that this function is well-defined, one-to-one, and not onto.

For g we choose $g(x) = x$. Other answers are possible.

(b) For f we define $f(x) = |\lfloor x \rfloor|$. (The inside function is the floor function, also called the greatest integer function. It is defined by $\lfloor x \rfloor = \max\{n \in \mathbb{N} : n \leq x\}$.)

For g list the positive rationals as follows.

$$\begin{array}{cccccc} 0 & 1 & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\ & \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \cdots \\ & \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\ & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

We define g on $2\mathbb{N}$ by counting the entries of the table in a diagonal way from top right to bottom left: $g(0) = 0, g(2) = 1, g(4) = 2/1, g(6) = 1/2, g(8) = 3/1, g(10) = 2/2$, etc.

On the negative integers we go through the list in the same way starting with 1 and multiplying each value by -1: $g(1) = -1, g(3) = -2/1, g(5) = -1/2, g(7) = -3/1, g(9) = -2/2$, etc.

(c) The existence of such a function would imply that $|\mathbb{R}| \leq |\mathbb{Q}|$. By Theorem 22.11, $|\mathbb{Q}| = \aleph_0$ and thus we would have a contraction. Hence no such function exists.

(d) Such a function is not possible either. If it were, then we would have $|\mathcal{P}(\mathbb{R})| \leq |\mathbb{R}|$. By Cantor's Theorem, $|\mathbb{R}| < |\mathcal{P}(\mathbb{R})|$. The two inequalities contradict each other.

Solution to Problem 24.12. *(a) The five sequences are as follows.*

$$\begin{array}{l} (0,0,0,\ldots) \\ (1) \\ (45,15,5) \\ \big(4(3^6), 4(3^5), 4(3^4), 4(3^3), 4(3^2), 4(3), 4, -1\big) \\ \big(5(3^6), 5(3^5), 5(3^4), 5(3^3), 5(3^2), 5(3), 5\big) \end{array}$$

(b) We will find $\mathbb{N}_a$, $\mathbb{N}_\infty$ and, while we are at it, we find $\mathbb{N}_b$. (Note that $\mathbb{N}_b$ was not requested.) We can write every $n \in \mathbb{N}$ as $n = k(3^m)$, where k and m are natural numbers with $3 \not| k$. Now we consider all possible cases.

Case 1 *If $k = 0$, then $n = 0 \in \mathbb{N}_\infty$ as seen in part (a).*

Case 2 *If $k > 0$ and m is odd, then the $(m+1)$st term is k and is in $\mathbb{Z}$. Since $k \notin$ range(f), the sequence stops and is of even length.*

Case 3 *If $k = 1$ and m is even, then the $(m+1)$st term is 1 and is in $\mathbb{N}$. Since $1 \notin$ range(g), the sequence stops and is of odd length.*

Case 4 *If $k > 1$, $k \equiv 1 \mod 3$, and m is even, then the $(m+1)$st term is $k = 3s+1$ for some $s \geq 1$ and it is in $\mathbb{N}$. It follows that the $(m+2)$nd term is $-s$ in $\mathbb{Z}$. Now $-s \notin$ range(f), the sequence stops and is of even length.*

Case 5 *If $k \equiv 2 \mod 3$ and m is even, then the $(m+1)$st term is $k = 3t+2$ for some $t \geq 0$ and it is in $\mathbb{N}$. Since $k = 3t+2 \notin$ range(g), the sequence stops and it is of odd length.*

This exhausts all possibilities and we just need to collect the elements for each set.

$\mathbb{N}_\infty = \{0\}$.

$\mathbb{N}_a = \{3^{2s} : s \in \mathbb{N}\} \cup \{k(3^{2s}) : k, s \in \mathbb{N} \text{ and } k \equiv 2 \ (mod\ 3)\}$

$\mathbb{N}_b = \{k(3^{2s+1}) : k, s \in \mathbb{N} \text{ and } k \geq 1\} \cup \{k(3^{2s}) : k, s \in \mathbb{N}, k > 1, \text{ and } k \equiv 1 \, (mod\ 3)\}$

Solution to Problem 24.15. *Clearly the function $f : (0,1) \longrightarrow (0,1) \times (0,1)$ defined by $f(x) = (x,x)$ is an injection and thus $|(0,1)| \leq |(0,1) \times (0,1)|$.*

To construct an injection in the other direction, first note that every $x \in (0,1)$ can be written using a decimal representation $x = 0.x_1x_2x_3x_4\ldots$. Here $x_j \in \{0,1,\ldots,9\}$ denotes the j-th digit. This representation is unique if we agree to avoid sequences of digits that end in all 9's. Thus, if $x = 0.x_1x_2x_3x_4\ldots$, then $x_k \neq 9$ and $x_j = 9$ for all $j > k$. Then we will write $x = 0.x_1x_2x_3x_4\ldots(x_k+1)$. (Convince yourself that this is the same number!)

We now define $g : (0,1) \times (0,1) \longrightarrow (0,1)$ by the following scheme. For $(x,y) \in (0,1) \times (0,1)$ we use the unique decimal representations $x = 0.x_1x_2x_3x_4\ldots$ and $y = 0.y_1y_2y_3y_4\ldots$ (avoiding tails of 9's) and define $g(x,y) = 0.x_1y_1x_2y_2x_3y_3\ldots$.

Given $(x,y) \in (0,1) \times (0,1)$, we get $g(x,y) \in (0,1)$ and this value is unique, showing that the function is well-defined. Suppose $(x,y),(w,z) \in (0,1) \times (0,1)$ and $(x,y) \neq (w,z)$. Without loss of generality, we may assume that $x \neq w$. Then there is some integer k such that $x_k \neq w_k$. This implies that the $2k-1$st digit of $g(x,y)$ is x_k and the $2k-1$st digit of $g(w,z)$ is w_k. Since none of x, y, w, and z has a tail consisting of 9's neither do $g(x,y)$ and $g(w,z)$. That means the decimal representations are unique and thus $g(x,y) \neq g(w,z)$. Hence g is injective and $|(0,1) \times (0,1)| \leq |(0,1)|$.

Using Cantor-Schröder-Bernstein we conclude that $|(0,1) \times (0,1)| = |(0,1)|$.

Solution to Problem 24.18. *We define* $g : 2^X \longrightarrow \mathcal{P}(X)$ *by* $g(f) = \{x \in X : f(x) = 1\}$.
Given $f \in 2^X$, *the set* $g(f) \subseteq X$. *Thus to every* $f \in 2^X$ *there is* $g(f) \in \mathcal{P}(X)$. *Suppose that* $g(f) = A$ *and* $g(f) = B$. *If* $x \in A$ *then* $f(x) = 1$. *Thus,* $x \in B$. *The converse uses the same argument. Hence* $A = B$, *which shows that* g *is a well-defined function.*

Suppose that $g(f_1) = g(f_2)$ *and let* $x \in X$. *If* $f_1(x) = 1$, *then* $x \in g(f_1)$. *Hence* $x \in g(f_2)$. *We conclude that* $f_2(x) = 1$. *If* $f_1(x) = 0$, *then* $x \notin g(f_1)$. *Hence* $x \notin g(f_2)$. *We conclude that* $f_2(x) = 0$. *Thus* f_1 *and* f_2 *have the same domain and range and assign the same value to each of the elements in the domain. So,* $f_1 = f_2$ *and the function* g *is injective.*

Let $A \in \mathcal{P}(X)$. *We let* $f : X \longrightarrow \{0, 1\}$ *defined by* $f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ *By definition of* g *we have* $g(f) = A$. *Thus* g *is onto.*

We have shown that g *is a bijection and thus* $|\mathcal{P}(X)| = |2^X|$. *(Note that we are in fact assigning to each* $A \in \mathcal{P}(X)$ *its characteristic function* $\chi_A \in 2^X$.*)*