Reading, Writing, and Proving (Second Edition)

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Springer Verlag, 2011

Solutions to Chapter 23: Countable and Uncountable Sets

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Solution to Problem 23.3. *(a) A line with a rational slope is uniquely determined by its slope m and its y-intercept b. Hence the set of all lines with rational slopes is equivalent to $\{(m,b) : m \in \mathbb{Q}, b \in \mathbb{R}\} = \mathbb{Q} \times \mathbb{R}$. Since $\mathbb{R}$ is uncountable and $\mathbb{R} \approx (\{0\} \times \mathbb{R}) \subseteq \mathbb{Q} \times \mathbb{R}$, Corollary 22.4 implies that the set of all lines with rational slopes is uncountable.*

(b) Since $\mathbb{Q} = (\mathbb{Q} \setminus \{0\}) \cup \{0\}$ we conclude that $\mathbb{Q} \setminus \{0\}$ is countably infinite.

(c) Since $\mathbb{N}$ is countably infinite and $\{1,3\}$ is finite, $\mathbb{N} \setminus \{1,3\}$ is countably infinite.

(d) We can define a function $f : \mathbb{R} \to \{(x,y) \in \mathbb{R} \times \mathbb{R} : x + y = 1\}$ by $f(x) = (x, 1-x)$. One can check that this is well-defined and bijective. Hence $\{(x,y) \in \mathbb{R} \times \mathbb{R} : x + y = 1\} \approx \mathbb{R}$ and so this set is uncountable.

(e) In the proof of Theorem 22.12 we showed that $(0,1)$ is uncountable. Since $(0,1) \subseteq [0,\infty)$, Corollary 22.4 implies that $[0,\infty)$ is uncountable.

Solution to Problem 23.6. *Many of the details below appear in this chapter or previous ones. We provide the basics below.*

(a) $\mathbb{Z}$ is equivalent to $\mathbb{N}$ and $\mathbb{N} \subset \mathbb{Z}$.

(b) If A is countably infinite, then there is a bijection $f : \mathbb{N} \to A$. Consider $A_1 = A \setminus \{f(0)\}$. Then $f|\mathbb{Z}^+$ is a bijection of $\mathbb{Z}^+$ onto A_1. Since $\mathbb{Z}^+$ and $\mathbb{N}$ are equivalent, there is a bijection g from $\mathbb{Z}^+$ onto $\mathbb{N}$. Therefore $f \circ g \circ (f|\mathbb{Z}^+)^{-1}$ is a bijection from A_1 onto A. Since $A_1 \subset A$, this completes the example.

(c) If A is uncountable, we may choose any $a \in A$ and consider the set $A_a = A \setminus \{a\}$. You should provide the details that A_a must be uncountable.

(d) We have $B \subseteq A$ and $|A| = |B|$. Therefore, Problem 22.18 part (c) implies that $A = B$.

Solution to Problem 23.9. *We know that every subset of* $\mathbb{N}$ *is countable and we must show that every subset of a countable set is countable. We have already seen that every subset of a finite set is finite, so we assume that our set,* A*, is infinite. Therefore, there is a bijection* $f : A \to \mathbb{N}$*. Let* A_1 *be a subset of* A*. If* $A_1 = \emptyset$*, then it is countable, so assume* $A_1 \neq \emptyset$*. Then* $f|A_1$ *maps* A_1 *onto a subset* S *of* $\mathbb{N}$*. We have already seen that the fact that* f *is one-to-one implies* $f|A_1$ *is one-to-one. Therefore,* $f|A_1$ *is a bijection between* A_1 *and a countable set* S*. Hence* A_1 *is countable.*

Solution to Problem 23.12. *We think of the numbers as forming an infinite square matrix. Define* $f(0)$ *to be the* $(1, 1)$ *entry. Thus,* $f(0) = 1$*. Now we define* f *moving from top to bottom along the diagonal (skipping the fractions we have already defined) as follows:*
define $f(1) = 2/1$*;*
define $f(2) = 1/2$*;*
define $f(3) = 3/1$*;*
define $f(4) = 1/3$*, etc.*
Then f *will be a bijection.*

Solution to Problem 23.15. *For* $x \in (0, 1)$*, denote the decimal expansion by* $x = 0.x_1x_2x_3\ldots$*, where* $x_j \in \{0, 1, \ldots, 9\}$*. This expansion is unique if we decide to replace every sequence that is finite and ends in* 1 *with the infinite sequence ending in a string of* 9*'s (see the paragraph before Theorem 22.12 of the text). Define the function* $f : (0, 1) \to \mathbb{N}^\infty$ *by* $f(x) = (x_1, x_2, x_3, \ldots)$*. Because of the uniqueness of the decimal expansion, this is a well-defined function. It is clearly one-to-one. Thus* $(0, 1) \approx ran(f) \subseteq \mathbb{N}^\infty$*. Since* $(0, 1)$ *is uncountable, Corollary 22.4 implies that* $\mathbb{N}^\infty$ *is uncountable.*

Solution to Problem 23.18. *We define* $f : \mathcal{P}(A \cup B) \to \mathcal{P}(A) \times \mathcal{P}(B)$ *by* $f(X) = (X \cap A, X \cap B)$*. First we show that* f *is well-defined. A domain and codomain are specified. For any* $X \in \mathcal{P}(A \cup B)$ *we have* $X \cap A \in \mathcal{P}(A)$ *and* $X \cap B \in \mathcal{P}(B)$*. Thus* $f(X)$ *is defined as an element in* $\mathcal{P}(A) \times \mathcal{P}(B)$*. Suppose now that for some* $X \in \mathcal{P}(A \cup B)$ *we have* $f(X) = (U, V)$ *and* $f(X) = (W, Z)$*. Then* $U = X \cap A$ *and* $W = X \cap A$*. Thus* $U = W$*. Similarly,* $V = Z$*. Thus* $(U, V) = (W, Z)$*. This shows that the function is well-defined.*

Next we show that f *is injective. So suppose that for* $W, Z \in \mathcal{P}(A \cup B)$ *we have* $f(W) = f(Z)$*; that is,* $W \cap A = Z \cap A$ *and* $W \cap B = Z \cap B$*. Note that* $W \subseteq A \cup B$ *and* $Z \subseteq A \cup B$*. Thus, if* $x \in W$*, then* $x \in A \cup B$*. Thus* $x \in A$ *or* $x \in B$*. If* $x \in A$*, then* $x \in W \cap A = Z \cap A$*. Thus* $x \in Z$*. If* $x \in B$*, then* $x \in W \cap B = Z \cap B$*. Thus again* $x \in Z$*. This shows that* $W \subseteq Z$*. Reversing the roles of* W *and* Z *shows that* $W \subseteq Z$*. Thus* $W = Z$*. This shows that* f *is injective.*

Finally we will show that f *is surjective. Let* $(U, V) \in \mathcal{P}(A) \times \mathcal{P}(B)$*. Then* $U \subseteq A$ *and* $V \subseteq B$*. Set* $X = U \cup V$*. Then* $X \subseteq A \cup B$ *and thus* $X \in dom(f)$*. Now* $f(X) = (X \cap A, X \cap B)$*. We claim that* $X \cap A = U$*. If* $x \in X \cap A$ *then* $x \in X = U \cup V$ *and* $x \in A$*. If* $x \in V \subseteq B$*, then* $x \notin A$ *because* $A \cap B = \emptyset$*. This is not possible, hence* $x \in U$*. We have shown that* $X \cap A \subseteq U$*. If* $x \in U \subseteq A$*, then also* $x \in A$*. Since* $U \subseteq U \cup V = X$ *we also have* $x \in X$*. Thus* $x \in X \cap U$*. Hence* $U \subseteq X \cap A$*. This establishes the claim. In the same way we show that* $X \cap B = V$*. Thus* $f(X) = (U, V)$ *and we have shown that* f *is surjective.*

We have now established a bijection $f : \mathcal{P}(A \cup B) \to \mathcal{P}(A) \times \mathcal{P}(B)$*. This proves that* $\mathcal{P}(A \cup B) \approx \mathcal{P}(A) \times \mathcal{P}(B)$*.*

Solution to Problem 23.21. *For our definition, f is decreasing if $x \leq y$ implies $f(x) \geq f(y)$. Some authors use decreasing as another way of describing "strictly decreasing." That will change this solution slightly, but the main idea is the same.*

Note that the well-ordering principle implies that every decreasing function from $\mathbb{N}$ to $\mathbb{N}$ is eventually constant. Therefore, we may think of this set as a subset of the set of all eventually constant sequences. (Think about this before proceeding!) Let E denote the set of eventually constant sequences (where each sequence is of the form $(x_n)_{n\geq 1}$). Define a map $F : E \to \mathbb{Z}$ by

$$F((x_n)) = 2^{x_1}3^{x_2}5^{x_3}\cdots p_m^{x_m},$$

where x_m is the first term in the sequence for which $x_m = x_n$ for all $n \geq m$ and $2, 3, \ldots, p_m$ are the prime numbers listed in increasing order. Then you should check that F defines a one-to-one map of E into the countable set $\mathbb{Z}$. The desired conclusion follows from this.

If you choose to use the fact that a countable union of countable sets is countable, there is another way to prove this. (See Projects 29.12 and Theorem 29.13.)

Let A_0 denote the set of all constant sequences, A_1 the set of all sequences that are constant after the first term, and, in general, let A_n denote the set of all sequences that are constant after the n-th term. We will show that A_n is countable for each n.

Since $\mathbb{N}$ is countable, the map $f : \mathbb{N} \to A_0$ defined by $f(n) = (n, n, n, \ldots)$ defines a bijection between $\mathbb{N}$ and A_0. Hence A_0 is countable.

In general, for A_n, we know that $B_n = \mathbb{N} \times \cdots \mathbb{N}$ (taken n times) is countably infinite. If we let g be a bijective map from $\mathbb{N}$ onto B we may define $h : \mathbb{N} \times \mathbb{N} \to A_n$ by $h(n, m) = (g(n), m, m, \ldots)$. It is not difficult to check that h is a bijection. Since the domain is the Cartesian product of two countable sets Corollary 23.10 implies that the domain is countable. So the domain (which is obviously infinite) is countably infinite. Therefore A_n is countable for each $n \in \mathbb{N}^+$. The set we are interested in is equivalent a subset of $\cup_n A_n$ and a countable union of countable sets is countable, we have the desired result.