Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 22: Finite Sets and an Infinite Set

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Solution to Problem 22.3. *Suppose to the contrary that no integers m and n are among the twenty one that are chosen from the set $\{1, \ldots, 40\}$ so that $n - m = 1$. That means there are no consecutive numbers among the twenty one chosen. Hence the difference between the largest and the smallest is at least 40. This is a contradiction, since the integers are chosen from the set $\{1, \ldots, 40\}$ and, therefore, the difference can be at most 39. We conclude that there are two numbers m, n such that $n - m = 1$.*

Solution to Problem 22.6. *Suppose to the contrary that there is an injective function $f : \{1, \ldots, 99\} \to \{1, \ldots, 99\}$ such that $g : \{1, \ldots, 99\} \to \mathbb{N}$ defined by $g(n) = n + f(n)$ has the property that $g(n)$ is odd for all $n \in \{1, \ldots, 99\}$. This implies that if n is odd, then $f(n)$ is even.*
We define the following functions: $h : \{1, \ldots, 50\} \to \{1, \ldots, 99\}$ by $h(n) = 2n - 1$ and $j : \{2, 4, 6, \ldots, 98\} \to \{1, \ldots, 49\}$ by $j(m) = m/2$. We note that h and j are both injective and that $ran(h) \subseteq dom(f)$. Since n odd implies that $f(n)$ is even, we have $ran(f \circ h) \subseteq dom(j)$. Thus we can compose

$$(j \circ f \circ h) : \{1, \ldots, 50\} \to \{1, \ldots, 49\}$$

Since each function is injective Theorem 5.7 implies that $j \circ f \circ h$ is injective. This contradicts Theorem 21.2 (the pigeonhole principle).
We conclude that for some integer n, the value of $g(n)$ is even.

Solution to Problem 22.9. *Proof.* Suppose to the contrary that $\mathbb{R}$ is finite. Since $\mathbb{R} \neq \emptyset$, there must exist a positive integer n such that $\mathbb{R} \approx \{1, \ldots, n\}$. That is, there is a bijective function $f : \mathbb{R} \to \{1, \ldots, n\}$. Then $f|_{\{1,\ldots,n+1\}} : \{1, \ldots, n+1\} \to \{1, \ldots, n\}$ is injective, since it is the restriction of an injective function. This contradicts the pigeonhole principle. Thus $\mathbb{R}$ is infinite. □

Solution to Problem 22.12. (a) *Note that $A \cap B \subseteq A$. Since A is finite, Corollary 21.10 implies that $A \cap B$ is finite.*

(b) *We also have* $A \setminus B \subseteq A$*. Using Corollary 21.10 again, we conclude that* $A \setminus B$ *is finite.*

(c) *We claim that* $X \setminus A$ *is infinite. Suppose that this set were finite. Since* $X = (X \setminus A) \cup A$*, Theorem 21.11 would imply that* X *is finite. This is a contradiction and shows that* $X \setminus A$ *is infinite.*

(d) *Theorem 21.11 shows that* $A \cup B$ *is finite.*

(e) *We define* $g : A \to f(A)$ *by* $g(x) = f(x)$*. This is a well-defined function that is one-to-one and surjective. Hence* $f(A) \approx A$*. Thus* $f(A)$ *is finite.*

Solution to Problem 22.15. *If* $X = \emptyset$ *then* $|X| = 0$ *and* $\mathcal{P}(X) = \{\emptyset\}$*. Hence* $|\mathcal{P}(X)| = 1 = 2^0$ *and the formula holds.*

Now consider the case when $X \neq \emptyset$*, so* $n \geq 1$*. We define the set of all sequences of length* n *with terms either 0 or 1:* $Y = \{(x_m) : x_m \in \{0,1\} \text{ for } 1 \leq m \leq n\}$*. For each sequence we have exactly two choices at each of the* n *places. Thus, there are* 2^n *different sequences. This implies that* $|Y| = 2^n$*. We enumerate the elements of* X *and write* $X = \{a_1, \dots, a_n\}$ *Now define* $f : Y \longrightarrow \mathcal{P}(X)$ *by* $f((x_m)) = \{a_j \in X : \text{ for all } j \text{ with } x_j = 1\}$*. This is a well-defined function that can be shown to be bijective, hence* $|\mathcal{P}(X)| = |Y| = 2^n$*.*

Solution to Problem 22.18. (a) *If* $B = \emptyset$*, then* $|B| = 0 \leq |A|$ *for any finite set* A*. So assume now that* $B \neq \emptyset$*. Then* $A \neq \emptyset$*. Then there exist positive integers* m *and* n *and bijective functions* $f : \{1, \dots, m\} \to B$ *and* $g : \{1, \dots, n\} \to A$*. We denote by* h *the inclusion* $h : B \to A$*, defined by* $h(x) = x$*. We consider the composition* $g^{-1} \circ h \circ f : \{1, \dots, m\} \to \{1, \dots, n\}$*. Since each function is injective, so is the composition. The pigeonhole principle implies that* $m \leq n$*. Thus* $|B| = m \leq n = |A|$*.*

(b) *If* B *is strictly contained in* A*, then there exists* $a \in A$ *with* $a \notin B$*. Thus* $B \subseteq A \setminus \{a\}$*. Using the results of part (a) and Problem 22.17 we get*

$$|B| \leq |A \setminus \{a\}| = |A| - 1 < |A|.$$

Hence $|B| < |A|$*.*

(c) *Suppose to the contrary that* $B \neq A$*. By part (b) of this problem we conclude that* $|B| < |A|$*. This contradicts the assumption that* $|A| \leq |B|$*. Hence we must have* $A = B$*.*

Solution to Problem 22.21. *Note that* $f(A) \subseteq A$ *and thus* $|f(A)| \leq |A|$ *as shown in Problem 22.18 (a).*

Assume that f *is injective. Then* $|A| = |f(A)|$*. The contrapositive statement of Problem 22.18 shows that* $A = f(A)$*. Hence* f *is surjective.*

Conversely, assume that f *is surjective. Then* $f(A) = A$*. Suppose to the contrary that* f *is not injective. Then there are* $a, b \in A$*,* $a \neq b$*, and* $f(a) = f(b)$*. We define* $g : A \setminus \{a\} \to A$ *by* $g(x) = f(x)$ *(the restriction of* f *to* $A \setminus \{a\}$*). By Problem 22.20 we conclude that* $|A| = |f(A)| = |g(A \setminus \{a\})| \leq |A \setminus \{a\}|$*. Using the* *result of Problem 22.17, we conclude that* $|A| \leq |A \setminus \{a\}| = |A| - 1$. *This is a contradiction, hence* f *is injective.*

This is not true if the set A *is infinite. Check that* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = e^x$ *is injective but not surjective. On the other hand, the function* $g : \mathbb{R} \to \mathbb{R}$ *defined by* $g(x) = x(x-1)(x-2)$ *is surjective but not injective.*