Reading, Writing, and Proving (Second Edition)

Ulrich Daepp and Pamela Gorkin

Springer Verlag, 2011

Solutions to Chapter 21: Equivalent Sets

A Note to Student Users. Check with your instructor before using these solutions. If you are expected to work without any help, do not use them. If your instructor allows you to find help here, then we give you permission to use our solutions provided you credit us properly.

If you discover errors in these solutions or feel you have a better solution, please write to us at udaepp@bucknell.edu or pgorkin@bucknell.edu. We hope that you have fun with these problems.

Ueli Daepp and Pam Gorkin

Solution to Problem 21.3. *A "finite union of sets" is the set of all elements that are contained in at least one of a finite family of sets. The sets of this family may be finite or infinite but there are only finitely many of them.*
A " union of finite sets" is the set of all elements that are contained in at least one of a family of sets and all sets in this family are finite. There is nothing said about whether the family consists of finitely or infinitely many sets.

Solution to Problem 21.6. *(a) We define $f : (0,1) \to \mathbb{R}$ by $f(x) = \tan(\pi(x - 1/2))$. We know, from calculus, that $f'(x) = \pi \sec^2(\pi(x - 1/2)))$. Thus for $0 < x < 1$, the derivative $f'(x) > 0$. We conclude that f is injective.*

We also know that $\lim_{x\to 0} \tan(\pi(x - 1/2)) = -\infty$, $\lim_{x\to 1} \tan(\pi(x - 1/2)) = \infty$, and that $f(x) = \tan(\pi(x - 1/2))$ is continuous on the interval $(0, 1)$. By the intermediate value theorem, the function is surjective.

We have shown that $(0, 1) \approx \mathbb{R}$. (For a proof without calculus, use the function from Problem 15.13.)

(b) Using the function f from part (a) and restricting it to $(1/2, 1)$ yields a bijection $f|_{(1/2,1)} : (1/2, 1) \to \mathbb{R}^+$. We define $g : (0, 1) \to (1/2, 1)$ by $g(x) = (x + 1)/2$. The function g is a bijection (show this). We can now compose these functions to get a new function $h = f|_{(1/2,1)} \circ g \circ f^{-1}$. Since h is the composition of bijections, it is again a bijection and you check that $h : \mathbb{R} \to \mathbb{R}^+$. Hence $\mathbb{R} \approx \mathbb{R}^+$.

Solution to Problem 21.9. *(a) Since A and B are nonempty finite sets, there are positive integers n and m and bijections $f : A \to \{1, \ldots, n\}$ and $g : B \to \{1, \ldots, m\}$. We conclude immediately that $A \approx \{1, \ldots, n\}$. We define $h : \{1, \ldots, m\} \to \{n + 1, \ldots, n + m\}$ by $h(x) = n + x$. You should check that h is a well-defined bijective function. Hence the composition $h \circ g : B \to \{n + 1, \ldots, n + m\}$ is a bijection. Thus, $B \approx \{n + 1, \ldots, n + m\}$.*

(b) *Proof.* If $A = \emptyset$ or $B = \emptyset$, then $A \cup B$ is trivially finite. So assume now that A and B are both finite nonempty sets with $A \cap B = \emptyset$. We also have $\{1, \ldots, n\} \cap \{n+1, \ldots, n+m\} = \emptyset$. By part (a) of this problem, $A \approx \{1, \ldots, n\}$ and $B \approx \{n+1, \ldots, n+m\}$. It follows from Theorem 21.6 that $A \cup B \approx \{1, \ldots, n\} \cup \{n+1, \ldots, n+m\} = \{1, \ldots, n+m\}$. In particular, this shows that $A \cup B$ is finite. □

Solution to Problem 21.12. (a) *Proof.* We define $f : \mathcal{A} \to \mathbb{R} \times \mathbb{R}^+$ by $f([a, b)) = (a, b - a)$. Since $a < b$, we have $b - a > 0$, so this is a well-defined function.

To show that f is injective, suppose that $f([a, b)) = f([c, d)) = (x, y)$. Then $x = a$ and $x = c$. So $a = c$. Also, $b - a = y = d - c$. This implies that $b = d$. Hence $[a, b) = [c, d)$. Thus, f is injective.

To show that f is surjective, let $(x, y) \in \mathbb{R} \times \mathbb{R}^+ = cod(f)$. Then $x, y \in \mathbb{R}$ and $x < x + y$. Hence $[x, x + y) \in \mathcal{A} = \text{dom}(f)$. We find $f([x, x + y)) = (x, y)$. Thus f is surjective.

Since we have a bijective function between the two sets, we conclude that $\mathcal{A} \approx \mathbb{R} \times \mathbb{R}^+$. □

(b) *Proof.* In Problem 21.6 Part (b) you showed that $\mathbb{R} \approx \mathbb{R}^+$. It follows from Theorem 21.13 (proven in Problem 21.10) that $\mathbb{R} \times \mathbb{R} \approx \mathbb{R} \times \mathbb{R}^+$. By symmetry and transitivity of set equivalence we conclude that $\mathcal{A} \approx \mathbb{R} \times \mathbb{R}$. □

Solution to Problem 21.15. *First we note that $f : \mathbb{Z}^+ \to \mathbb{N}$ defined by $f(x) = x - 1$ is a bijective map. Hence $\mathbb{Z}^+ \approx \mathbb{N}$. By Theorem 20.5, $\mathbb{N} \approx \mathbb{Z}$. Using Theorem 20.13 we have $\mathbb{Z}^+ \times \mathbb{N} \approx \mathbb{N} \times \mathbb{Z}$. Using the same arguments we get $\mathbb{Z}^- \times \mathbb{N} \approx \mathbb{Z}^- \times \mathbb{Z}$. Note that $(\mathbb{Z}^+ \times \mathbb{N}) \cap (\mathbb{Z}^- \times \mathbb{N}) = \emptyset$ and that $(\mathbb{N} \times \mathbb{Z}) \cap (\mathbb{Z}^- \times \mathbb{Z}) = \emptyset$. Thus, we can apply Theorem 20.6 and get $(\mathbb{Z}^+ \times \mathbb{N}) \cup (\mathbb{Z}^- \times \mathbb{N}) \approx (\mathbb{N} \times \mathbb{Z}) \cup (\mathbb{Z}^- \times \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$. Hence $(\mathbb{Z}^+ \times \mathbb{N}) \cup (\mathbb{Z}^- \times \mathbb{N}) \approx \mathbb{Z} \times \mathbb{Z}$.*

Solution to Problem 21.18. *We define a function $f : \mathcal{P}(\{1, 2, 3, 4, 5\}) \to A$ by*

$$f(X) = (a_1, a_2, a_3, a_4, a_5), \quad \textit{where } a_j = \begin{cases} 1 & \textit{if } j \in X \\ 0 & \textit{if } j \notin X \end{cases}.$$

We will first show that f is well-defined. A domain and codomain are specified. Let $X \in \mathcal{P}(\{1, 2, 3, 4, 5\})$. For each j with $1 \le j \le 5$ we have $a_j = 1$, if $j \in X$, or $a_j = 0$, if $j \notin X$. Hence $(a_1, a_2, a_3, a_4, a_5) \in A$. Thus $f(X) \in A = cod(f)$ is defined. Now suppose that for $X \in \mathcal{P}(\{1, 2, 3, 4, 5\})$ with $f(X) = (a_1, a_2, a_3, a_4, a_5)$ and $f(X) = (b_1, b_2, b_3, b_4, b_5)$. For each $j, 1 \le j \le 5$, if $j \in X$, then $a_j = 1$ and $b_j = 1$. If $j \notin X$, then $a_j = 0$ and $b_j = 0$. Thus for all j, we get $a_j = b_j$. Hence $(a_1, a_2, a_3, a_4, a_5) = (b_1, b_2, b_3, b_4, b_5)$. This shows that f is well-defined.

We will show that f is injective. So let $X, Y \in \mathcal{P}(\{1, 2, 3, 4, 5\})$ with $f(X) = (a_1, a_2, a_3, a_4, a_5) = (b_1, b_2, b_3, b_4, b_5) = f(Y)$. If $j \in X$, then $a_j = 1$ and thus $b_j = 1$. This implies that $j \in Y$. Hence we have $X \subseteq Y$. Using the same argument, we can show that $Y \subseteq X$. Hence $X = Y$ and the function is injective.

We will show that f is surjective. Let $(a_1, a_2, a_3, a_4, a_5) \in A$. Define $X \in \mathcal{P}(\{1, 2, 3, 4, 5\})$ by $j \in X$ if and only if $a_j = 1$. Then $X \in dom(f)$ and $f(X) = (a_1, a_2, a_3, a_4, a_5)$. Hence f is surjective.

We have constructed a bijective function between $\mathcal{P}(\{1, 2, 3, 4, 5\})$ and A, hence the two sets are equivalent.

Solution to Problem 21.21. *We define* $f : [0,1] \to (0,1)$ *by*

$$f(x) = \begin{cases} \frac{1}{2} & \textit{if } x = 0 \\ \\ \frac{1}{m+2} & \textit{if } x = \frac{1}{m} \textit{ for some } m \in \mathbb{Z}^+ \\ \\ x & \textit{otherwise} \end{cases}$$

We will first show that f *is well-defined. A domain and codomain are specified. If* $x \in [0,1]$*, then* $x = 0, x = 1/m$ *for some* $m \in \mathbb{Z}^+$*, or* x *isn't one of these. In each of the three cases,* $f(x)$ *is defined and, in fact,* $f(x) \in (0,1) = cod(f)$*. This is the first condition on a function. Suppose now that* $f(x) = y$ *and* $f(x) = z$ *for some* $x \in [0,1]$*. If* $x = 0$*, then* $x \neq 1/m$ *for all* $m \in \mathbb{Z}^+$*. Thus* $y = f(x) = f(0) = 1/2$*. Likewise,* $z = 1/2$*. Hence* $y = z$*. If* $x = 1/m$ *for some* $m \in \mathbb{Z}^+$*, then* $x \neq 0$*. Hence* $y = f(x) = 1/(m+2)$*. Likewise,* $z = 1/(m+2)$*. Hence* $y = z$*. Finally, if* $x \neq 0$ *and* $x \neq 1/m$ *for all* $m \in \mathbb{Z}^+$*, then* $y = f(x) = x$ *and* $z = f(x) = x$*. Again we have* $y = z$*. This establishes the second condition for a function and shows that* f *is well-defined.*

We show that f *is one-to-one. Suppose that* $x, y \in [0,1]$ *and* $f(x) = f(y)$*. If* $x = 0$*, then* $f(x) = 1/2 = f(y)$*. If* $y = 1/m$ *for some* $m \in \mathbb{Z}^+$*, then* $f(y) = 1/(m+2) \neq 1/2$*. This is impossible. If* $y \neq 1/m$ *for some* $m \in \mathbb{Z}^+$ *and* $y \neq 0$*, then* $y \neq 1/2$ *and* $f(y) = y \neq 1/2$*. This is impossible. Hence* $y = 0 = x$*.*
If $x = 1/m$ *for some* $m \in \mathbb{Z}^+$*, then* $f(x) = 1/(m+2)$*. As above we can exclude the cases of* $y = 0$ *and* $y \neq 1/m$ *for all* $m \in \mathbb{Z}^+$*. Hence* $y = 1/n$ *for some* $n \in \mathbb{Z}^+$*. Then* $f(y) = 1/(n+2) = 1/(m+2)$*. It follows that* $m = n$ *and hence* $x = y$*.*
Finally, if $x \neq 0$ *and* $x \neq 1/m$ *for all* $m \in \mathbb{Z}^+$*, then* $f(x) = x$*. If* $y = 0$*, then* $f(y) = 1/2 \neq x$*. Hence, this case does not occur. If* $y = 1/m$ *for some* $m \in \mathbb{Z}^+$*, then* $f(y) = 1/(m+2) \neq x$ *since* $m + 2 \in \mathbb{Z}^+$*. Hence this is also impossible and we conclude that* $y \neq 0$ *and* $y \neq 1/m$ *for all* $m \in \mathbb{Z}^+$*. Then* $f(y) = y = x$*. Since* x *and* y *play symmetric roles we have considered all possible cases and have shown that* f *is injective.*

We need to show that f *is onto. Let* $y \in (0,1)$*. If* $y = 1/2$*, then* $f(0) = 1/2$ *and* $0 \in dom(f)$*. If* $y = 1/n$ *for some* $n \in \mathbb{Z}$ *with* $n \geq 3$*, set* $m = n - 2$*. Then* $m \in \mathbb{Z}^+$ *and thus* $1/m \in [0,1] = dom(f)$*. Further,* $f(1/m) = 1/(m+2) = 1/n = y$*. Finally, if* $y \neq 1/m$ *for all* $m \in \mathbb{Z}^+$*, then* $y \in (0,1) \subseteq dom(f)$ *and* $f(y) = y$*. This shows that* f *is surjective.*

We constructed a bijective function $f : [0,1] \to (0,1)$ *and, therefore, have shown that the two sets are equivalent.*