# Reading, Writing, and Proving (Second Edition)

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Springer Verlag, 2011

## Solutions to Chapter 20: Convergence of Sequences of Real Numbers



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If you discover errors in these solutions or feel you have a better solution, please write to us at udaepp@bucknell.edu or pgorkin@bucknell.edu. We hope that you have fun with these problems.

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**Solution to Problem 20.3.** *Proof.* From Problem 5.14 we conclude that $|x-a|<\delta$ if and only if $-\delta<x-a<\delta$. The latter inequalities are equivalent to $a-\delta<x<a+\delta$. This proves the statement. □

**Solution to Problem 20.6.** *(a) In Problem 18.5 (b) we found that* $S_1 = 75$ *and* $S_{n+1} = 0.65S_n + 75$. *Consider how the terms are formed:*

$$\begin{aligned} S_1 &= 75 \\ S_2 &= (0.65)75 + 75 = 75(1+0.65) \\ S_3 &= (0.65)75(1+0.65) + 75 = 75(1+0.65+0.65^2) \\ S_n &= 75(1+0.65+\ldots+0.65^{n-1}) \end{aligned}$$

*Using the formula for a geometric sum we guess that*

$$S_n = 75(1-0.65^n)/0.35 \ \textit{ for } n \geq 1.$$

*Proof.* We establish the claim with an inductive proof.

The case $n=1$: $S_1 = 75(1-0.65)/0.35 = 75$, which is correct.

Let $n \geq 1$ and suppose that $S_n = 75(1-0.65^n)/0.35$. Then

$$\begin{aligned} S_{n+1} &= 0.65S_n + 75 \\ &= 0.65\left(75\frac{1-0.65^n}{0.35}\right) + 75 \ \text{ (by induction hypothesis)} \\ &= 75\left(\frac{0.65-0.65^{n+1}}{0.35} + 1\right) \\ &= 75\left(\frac{1-0.65^{n+1}}{0.35}\right). \end{aligned}$$

□

(b) *We claim that* $\lim_{n\to\infty} S_n = 75/0.35$.

*Proof.* We first show that $(0.65)^n \to 0$.

To see this, let $\varepsilon > 0$. We may also assume $\varepsilon < 1$. Let $N = (\ln\varepsilon)/(\ln 0.65)$. Note that $N > 0$. For $n > N$ we have

$$\begin{aligned} |(0.65)^n - 0| &= (0.65)^n \\ &< (0.65)^N \\ &< \left(e^{\ln 0.65}\right)^{(\ln\varepsilon)/(\ln 0.65)} \\ &= \varepsilon \end{aligned}$$

This shows that $(0.65)^n \to 0$.

Using Theorem 20.9, we now see that $S_n \to 75/0.35$.

□

(c) $\lim_{n\to\infty} s_n = \lim_{n\to\infty}(S_n - 75) = \lim_{n\to\infty} S_n - 75 = \frac{75}{0.35} - 75 = \frac{975}{7}$.

(d) *After one week the amount of phenytoin in the patient's blood is between* $s_{14}mg$ *and* $S_{14}mg$*. This can be shown to be between* $138.77mg$ *and* $213.77mg$*.*

*After a full month, the level is between* $s_{60}$ *and* $S_{60}$*; that is, between* $139.29mg$ *and* $214.29mg$*.*

(e) *In the long run the patient's phenytoin level is between* $\lim_{n\to\infty} s_n$ *and* $\lim_{n\to\infty} S_n$*, which is between* $139.29mg$ *and* $214.29mg$*.*

*Note that there is no big difference, the phenytoin level is already stable after one week.*

**Solution to Problem 20.9.** (a) *We calculate our limit as follows:*

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{3n} &= \frac{1}{3}\lim_{n\to\infty}\frac{1}{n} \text{ (by Theorem 20.9 (ii))} \\ &= \frac{1}{3}\cdot 0 \text{ (by Example 20.2)} \\ &= 0. \end{aligned}$$

(c) *We note that the sequence* $(1/\sqrt{n})$ *is decreasing and bounded below. By Theorem 20.12 this sequence converges. So* $\lim_{n\to\infty}(1/\sqrt{n}) = a$ *for some real number* $a$*.*

*By Theorem 20.9 (iii), we have* $\left(\lim_{n\to\infty}(1/\sqrt{n})\right)^2 = a^2 = \lim_{n\to\infty}(1/n)$*. From Example 20.2 we conclude that* $a^2 = 0$*. Hence* $a = 0$*.*

*For all* $n \in \mathbb{Z}^+$*, we have* $0 < 1/\sqrt{n+7} < 1/\sqrt{n}$*. Problem 20.8 (a) implies that* $\lim_{n\to\infty}\frac{1}{\sqrt{n+7}} = 0$*.*

(d) *We calculate our limit as follows:*

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2+4}{n^2} &= \lim_{n\to\infty}\left(1+\frac{4}{n^2}\right) \\
&= \lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{4}{n^2} \text{ (by Theorem 20.9 (i))} \\
&= 1 + \left(\lim_{n\to\infty}\frac{2}{n}\right)^2 \text{ (by Theorem 20.9 (iii))} \\
&= 1 + \left(2\lim_{n\to\infty}\frac{1}{n}\right)^2 \text{ (by Theorem 20.9 (ii))} \\
&= 1 + 2\cdot 0 \text{ (by Example 20.2)} \\
&= 1.
\end{aligned}$$

(e) *Note that using Theorem 20.9 and Example 20.2 as above, we can show that* $\lim_{n\to\infty}\left(2+\frac{1}{n}\right) = 2$ *and* $\lim_{n\to\infty}\left(1+\frac{2}{n}\right) = 1$. *Then*

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{2n+1}{n+2}\right) &= \lim_{n\to\infty}\left(\frac{2+\frac{1}{n}}{1+\frac{2}{n}}\right) \\
&= \frac{\lim_{n\to\infty}\left(2+\frac{1}{n}\right)}{\lim_{n\to\infty}\left(2+\frac{1}{n}\right)} \text{ (by Theorem 20.9 (iii) and (iv))} \\
&= \frac{2}{1} = 2.
\end{aligned}$$

(g) *Note that* $0 < n < n+7 < (n+7)!$ *for all integers* $n \geq 1$. *Hence* $0 < \frac{1}{(n+7)!} < \frac{1}{n}$ *for all* $n \in \mathbb{Z}^+$. *By Example 20.2,* $\lim_{n\to\infty}\frac{1}{n} = 0$. *It follows from Problem 20.8 (a) that* $\lim_{n\to\infty}\frac{1}{(n+7)!} = 0$.

(h) *For this last part we leave it to you to fill in the details of the use of Theorem 20.9 and Example 20.2:*

$$\begin{aligned}
\lim_{n\to\infty}\frac{3n^2+1}{4n^2+n+2} &= \lim_{n\to\infty}\frac{3+\frac{1}{n^2}}{4+\frac{1}{n}+\frac{2}{n^2}} \\
&= \frac{\lim_{n\to\infty}\left(3+\frac{1}{n^2}\right)}{\lim_{n\to\infty}\left(4+\frac{1}{n}+\frac{2}{n^2}\right)} \\
&= \frac{3}{4}.
\end{aligned}$$

**Solution to Problem 20.12.** (a) *From Problem 5.14 we know that for all real numbers* $x_n$ *we have* $-|x_n| \leq x_n \leq |x_n|$. *Adding* $|x_n|$ *gives the required inequality*

$$0 \leq |x_n| + x_n \leq 2|x_n|.$$

(b) *As pointed out in part (a), we have* $-|x_n| \leq x_n \leq |x_n|$ *for all* $n$. *The result of Problem 20.8 (b) implies that* $x_n \to 0$.

(c) *The answer to this question is no. Consider* $x_n = (-1)^n$ *for* $n \in \mathbb{N}$. *Then* $|x_n| = 1$ *and thus* $|x_n| \to 1$. *In Exercise 20.6 you showed that* $(x_n)$ *does not converge.*

**Solution to Problem 20.15.** (a) *Since* $0 < a < 1$, *we have* $1 - a > 0$ *and* $x = (1-a)/a$ *is a positive real number. Then*

$$1/(1+x) = \frac{1}{1+\frac{1-a}{a}} = \frac{1}{\frac{a+1-a}{a}} = a.$$

(b) *Note that since* $x > 0$, *we have* $1 + x > 0$. *Bernoulli's inequality (Problem 18.6) applies and hence* $(1+x)^n \geq 1 + nx$ *for all* $n \in \mathbb{N}$. *Hence* $a^n = 1/(1+x)^n \leq 1/(1+nx)$ *for all* $n \in \mathbb{N}$.

(c) *Recall that* $x > 0$. *Hence for all* $n \geq 1$ *we have that*

$$0 < \frac{1}{1+nx} < \frac{1}{nx} = \left(\frac{1}{x}\right)\left(\frac{1}{n}\right).$$

*Using Example 20.2, Theorem 20.9 (ii), and Problem 20.8 (a) we conclude that* $1/(1+nx) \to 0$.

(d) *From Part (b) we have* $0 < a^n \leq 1/(1+nx)$ *for all* $n \in \mathbb{N}$. *Part (c) and Problem 20.8 (a) imply that* $a^n \to 0$.

(e) *Note that we can write* $x_n$ *alternatively as* $x_n = 1 - (1/10)^n$. *It now follows from Part (d) above and the rules of Theorem 20.9, that* $x_n \to 1$.

*Note that this justifies the equality* $0.99999\ldots = 1$ *and shows that the decimal representation of the real number 1 is not unique.*

**Solution to Problem 20.18.** *We have* $x_{n+1} = x_n + 9 \cdot 10^{-(n+1)} > x_n$ *and* $x_n < 1$ *for all* $n \in \mathbb{Z}^+$. *Hence* $(x_n)$ *is an increasing and bounded sequence. By Theorem 19.10 it converges to its supremum. In Problem 19.14 we showed that* $\lim_{n\to\infty} x_n = 1$. *Hence* $\sup(x_n) = 1$.

**Solution to Problem 20.21.** *We leave this one to you.*