Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 1:
The How, When, and Why of Mathematics

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If you discover errors in these solutions or feel you have a better solution, please write to us at udaepp@bucknell.edu or pgorkin@bucknell.edu. We hope that you have fun with these problems.

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Solution to Problem 1.3. *From The Merriam–Webster Dictionary, Merriam–Webster, Inc. Springfield, MA, 2004.* ***anagram*** *: a word or phrase made by transposing the letters of another word or phrase.*

(a) Vancouver

(b) Pennsylvania

(c) Philadelphia

(d) Dormitory

Solution to Problem 1.6. *As pointed out in Exercise 1.1, the letters occurring most frequently in an English text are: e, t, a , o, i, n, s, h, and r, in this order. Since our text is short, we cannot count on e occurring most frequently. But we try this strategy anyway, hoping that this short passage is somewhat representative of English writing. The letters that occur most frequently are:*

T: *5 times*

A: *4 times*

G: *4 times*

X: *4 times*

We will start by replacing T with E. We write down the two alphabets next to each other:

$$\begin{array}{ccc} A & \longrightarrow & L \\ B & \longrightarrow & M \\ C & \longrightarrow & N \\ & \ldots & \end{array}$$

This replacement yields:
CODING THEORY IS FUN. WE WILL LEARN MORE ABOUT IT LATER.

Punctuation and spaces were introduced to make complete sentences.

Is this the solution? We do not know for sure, but it is highly unlikely that any other shift would give a text that makes sense. We could try out all possible shifts, but we are willing to take the risk to claim that this is the only solution.

Solution to Problem 1.9. *1. "Understanding the problem." We recall from our earlier mathematical training that every integer is the unique product of prime numbers (up to the order of the factors). Since* $24 = 2^3 3$*, we need to show that* $n^3 - n$*, where* n *is odd, has* 3 *and* 2^3 *as factors.*

2. *"Devising a plan." The integer* $n^3 - n$ *can be factored and we hope to be able to show that* 3 *and* 2 *(three times) are prime factors of the factors.*

3. *"Carrying out the plan." We note that*

$$\begin{aligned} n^3 - n &= n(n^2 - 1) \\ &= n(n-1)(n+1) \\ &= (n-1)n(n+1) \end{aligned}$$

We found that $n^3 - n$ *is the product of three consecutive integers with the middle one being odd. Hence* $n - 1$ *and* $n + 1$ *are two consecutive even integers. One of them must be divisible by* 2 *and the other one by* 4*. Hence* $n^3 - n$ *is divisible by* 2^3*.*

Also, of three consecutive integers, one must be divisible by 3*. Hence* $n^3 - n$ *is divisible by* 3*.*

We conclude that $2^3 3$ *are prime factors of* $n^3 - n$*. Hence* $n^3 - n$ *is divisible by* 24*.*

Solution to Problem 1.12. *1. "Understanding the problem." What is the free throw shooting percentage? It is the ratio*

number of free throw baskets made / number of free throw baskets attempted.

It often helps to introduce some notation. Let n *= number of free throws attempted and* s *= number of free throws made. Then the free throw percentage of a player who had* n *attempts is*

$$p(n) = \frac{s}{n}.$$

The conditions on the problem can be stated as follows. There are positive integers m *and* n *such that* $m < n$ *and* $p(m) < 0.75 < p(n)$*. The question: Is there necessarily an integer* k *with* $m < k < n$*, such that* $p(k) = 0.75$*.*

2. *"Devising a plan." We can simplify the problem a little. In order to raise the free throw percentage from below 75% to above 75%, the player needs to have consecutive successful attempts. Thus we may assume that* $p(m) = s/m < 0.75 = 3/4$ *and* $p(n) = p(m + \ell) = (s + \ell)/(m + \ell) > 0.75 = 3/4$*.*

3. *"Carrying out the plan." We use the notation and the simplification introduced above.*

   Since $p(m+\ell) = (s+\ell)/(m+\ell) > 3/4$ *we conclude that* $4s + 4\ell > 3m + 3\ell$. *This implies that* $\ell > 3m - 4s$.

   Let $t = 3m - 4s$. *Then* $t < \ell$ *and*

$$\begin{aligned} p(m+t) &= \frac{s+t}{m+t} \\ &= \frac{s+3m-4s}{m+3m-4s} \\ &= \frac{3(m-s)}{4(m-s)} \\ &= \frac{3}{4} \end{aligned}$$

   Hence there exists $k = m + t = 4(m-s)$ *such that* $p(k) = 75\%$. *The question has an affirmative answer.*

4. *"Looking back." We believe that this is a rather surprising result. Certainly the free throw percentage increases by positive increments. Some percentages are skipped. Which ones can you not skip over? We can adapt our proof above to any free throw percentage that can be expressed as* $z/(z+1)$ *for a positive integer* z. *If a player is below such a percentage at the beginning of the season and above at the end, then at some point in the season, this is his or her exact free throw percenatage. In our case, 75% is 3/4, so* $z = 3$. *Write out the proof for a percentage of* $z/(z+1)$. *Follow the steps for the special case of* $z = 3$ *above. It will work.*

   What if the reference free throw percentage is not of the form $z/(z+1)$*? We will show that in this case, the exact percentage may not be attained. To this end, suppose that the free throw percentage to overtake in the season is* $s/(s+u)$ *for positive integers* s *and* u *satisfying* $u > 1$, *and that* s *and* $s+u$ *have no common divisor. Suppose further, that the player has a free throw percentage after* $s+u$ *attempts of*

$$p(s+u) = \frac{s-1}{s+u} < \frac{s}{s+u}.$$

   After that she will have ℓ *consecutive successful attempts until*

$$p(s+u+\ell) = \frac{s-1+\ell}{s+u+\ell} > \frac{s}{s+u}.$$

   We claim that there is no positive integer m *with* $m < \ell$ *such that* $p(s+u+m) = s/(s+u)$. *Suppose to the contrary that such a positive integer* m *exists. Then*

$$p(s+u+m) = \frac{s+m-1}{s+u+m} = \frac{s}{s+u}.$$

   Hence

$$\begin{aligned} s^2 + sm - s + us + um - u &= s^2 + us + ms \\ u(m-1) &= s \end{aligned}$$

Hence

$$\frac{s}{s+u} = \frac{u(m-1)}{u(m-1)+u} = \frac{m-1}{m}.$$

This is not possible because it would contradict the choice of s *and* u. *Hence the free throw percentage* $s/(s+u)$ *is skipped in this situation. This finishes our analysis.*

Note that this last part is rather difficult and you were not necessarily expected to come up with this complete solution to the extended problem. It shows an important aspect of mathematics. Almost every situation can be looked at in a more general way and often you can find and prove statements that go far beyond the original problem.