Reading, Writing, and Proving (Second Edition)

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Springer Verlag, 2011

Solutions to Chapter 19: Sequences

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Solution to Problem 19.3. *(a) We suggest* (x_n)*, where* $x_n = \frac{n^2+5}{n^2+4}$*. This sequence is bounded by* 2*, as you can check. Of course, there are many other examples.*

(b) Consider (y_n)*, defined by* $y_n = 3n - 7$*. This sequence has no upper bound. Since it is an increasing sequence,* $x_n \geq x_0 = -7$ *for all* $n \in \mathbb{N}$*. Hence* -7 *is a lower bound.*

(c) Consider (z_n)*, defined by* $z_n = 1 - \frac{1}{n+1}$*. This sequence is strictly increasing and* $\sup(z_n) = 1$*. However,* $z_n \neq 1$ *for all* $n \in \mathbb{N}$*.*

We claim that there does not exist a strictly increasing sequence that assumes its supremum.

Suppose to the contrary that there exists a sequence (w_n) *that is strictly increasing and that there exists* $m \in \mathbb{N}$ *such that* $x_m = \sup(w_n)$*. Then* $x_{m+1} > x_m = \sup(w_n)$*. This is a contradiction and the claim is proven.*

Solution to Problem 19.6. *We define* (x_n) *by*

$$x_n = \sum_{k=0}^{n} \frac{1}{k!}, \text{ for } n \in \mathbb{N}.$$

Clearly, $x_n \in \mathbb{Q}$ *for all* $n \in \mathbb{N}$ *and* (x_n) *is increasing. From calculus we recall the Taylor series of* e^x *and note that* $x_n \leq e^1$ *for all* n*. Thus* (x_n) *is increasing and bounded above, so we conclude that*

$$\sup(x_n) = \sum_{k=0}^{\infty} \frac{1}{k!} = e.$$

(A rigorous proof that the number e is irrational can be found in Project 29.5.)

Solution to Problem 19.9. *Our examples motivate us to make the following claim: If* $\sup(x_n) = \ell$, *then* $\inf(-x_n) = -\ell$.

Proof. Since $\ell = \sup(x_n)$, we have $\ell \geq x_n$ for all n (in the domain of the sequence). Hence $-\ell \leq -x_n$ for all n. This shows that $-\ell$ is a lower bound of $(-x_n)$.

Let u be a lower bound of $(-x_n)$. Then $u \leq -x_n$ for all n. Hence $-u \geq x_n$ for all n. Since ℓ is the supremum of (x_n), we conclude that $\ell \leq -u$. Hence $-\ell \geq u$. This completes the proof of the claim. □

Solution to Problem 19.12. *(a) Since* $\inf(x_n) \leq x_m$ *for all* $m \in \mathbb{N}$ *and* $\inf(y_n) \leq y_k$ *for all* $k \in \mathbb{N}$, *we conclude that for all* $\ell \in \mathbb{N}$ *we have* $\inf(x_n) + \inf(y_n) \leq (x_\ell + y_\ell)$. *This shows that* $\inf(x_n) + \inf(y_n)$ *is a lower bound of* $(x_n + y_n)$. *Thus* $\inf(x_n) + \inf(y_n) \leq \inf(x_n + y_n)$.

(b) We can have strict inequality. Consider (x_n) *defined by* $x_n = (-1)^n$ *and* (y_n) *defined by* $y_n = (-1)^{n+1}$. *Then* $x_n + y_n = 0$ *for all* $n \in \mathbb{N}$. *Hence*

$$\inf(x_n) + \inf(y_n) = -1 + (-1) = -2 < \inf(x_n + y_n) = 0.$$

Solution to Problem 19.15. *(a) Since* (x_n) *is bounded above, there exists* $M \in \mathbb{R}$ *such that* $x_n \leq M$ *for all* $n \in \mathbb{N}$. *This implies that for all* $n \in \mathbb{N}$ *we have* $y_n < x_{n+1} \leq M$. *Hence* (y_n) *is also bounded above. The completeness axiom of* $\mathbb{R}$ *implies that* $\sup(x_n)$ *and* $\sup(y_n)$ *both exist. We claim that* $\sup(x_n) = \sup(y_n)$. *From Problem 19.14 (b) we have that* $\sup(x_n) \leq \sup(y_n)$. *By our assumptions on the two sequences, we have* $y_n < x_{n+1} \leq \sup(x_n)$ *for all* $n \in \mathbb{N}$. *Thus* $\sup(x_n)$ *is an upper bound for* (y_n). *By the definition of the supremum for* (y_n), *we have* $\sup(y_n) \leq \sup(x_n)$. *This establishes the claim.*

(b) We claim that $\inf(x_n)$ *and* $\inf(y_n)$ *both exist and that* $\inf(x_n) < \inf(y_n)$.

Proof. The assumption implies that $x_n < x_{n+1}$ for all $n \in \mathbb{N}$. Hence (x_n) is a strictly increasing sequence and thus $inf(x_n) = x_0$.

Since $x_n < y_n$ for all n, we conclude that $x_{n+1} < y_{n+1}$ for all n. Hence $y_n < x_{n+1} < y_{n+1}$ for all $n \in \mathbb{N}$. Thus (y_n) is also strictly increasing and $\inf(y_n) = y_0$. Using the assumption for the special case of $n = 0$ we get $\inf(x_n) = x_0 < y_0 = \inf(y_n)$. □

Solution to Problem 19.18. *(a) We check that* $F_0 < F_1 \leq F_2 < F_3$, *since* $F_0 = 0, F_1 = 1, F_2 = 1$, *and* $F_3 = 2$. *We will further show that for* $n \geq 2$, *the Fibonacci sequence is strictly increasing. This will be done using the second principle of mathematical induction (Theorem 17.6). For the base step recall that* $F_2 = 1, F_3 = 2, F_4 = 3$, *and* $F_2 < F_3 < F_4$.

For the induction step, let $n \geq 3$ *and suppose that for all integers* m *with* $2 \leq m \leq n$ *we have* $F_{m+1} > F_m$. *Then, using the induction hypothesis, we get* $F_{n+2} = F_{n+1} + F_n > F_n + F_{n-1} = F_{n+1}$.

By induction, the sequence is strictly increasing for all $n \geq 2$ *and it is increasing for all* $n \in \mathbb{N}$.

(b) *We have $F_6 = 8$ and, as proven in part (a), F_n is a strictly increasing sequence of integers for $n \geq 6$. Thus $F_n > n$ for $n \geq 6$. (If this is not obvious, then you can prove it with induction.) That F_n is unbounded follows from Corollary 12.11.*

Solution to Problem 19.21. *Experimenting with the recursive definition leads us to the following claim: For all $n \in \mathbb{N}$, the function is defined by $f(n) = 2^{F_{n+1}}$, where F_k denotes the k-th term of the Fibonacci sequence.*

Proof. We will use induction to establish the claim.
For $n = 0$, we have $f(0) = 2 = 2^1 = 2^{F_1}$. For $n = 1$, we have $f(1) = 2 = 2^1 = 2^{F_2}$. Thus the formula is correct for $n = 0$ and $n = 1$.

Suppose that for some integer $n \geq 1$ and for all integers k, with $0 \leq k \leq n$, we know that $f(k) = 2^{F_{k+1}}$. Then

$$\begin{aligned} f(n+1) &= f(n)f(n-1) \text{ (by definition of } f) \\ &= 2^{F_{n+1}}2^{F_n} \text{ (by induction hypothesis)} \\ &= 2^{F_{n+1}+F_n} \\ &= 2^{F_{n+2}} \text{ (by the definition of the Fibonacci sequence).} \end{aligned}$$

The claim follows from the second principle of mathematical induction. □