Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 18: Mathematical Induction

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Solution to Problem 18.3. *We use induction on n.*

For the base step of $n = 1$, we check that $1^3 = 1^2$.

For the induction step we suppose that the formula is correct for some integer $n \geq 1$. We need to show that this implies that the formula is correct for $n + 1$. So

$$\begin{aligned}
(1+2+\cdots+n+(n+1))^2 &= (1+2+\cdots+n)^2+2(1+2+\cdots+n)(n+1)+(n+1)^2 \\
&= 1^3+2^3+\cdots+n^3+(n+1)(2(1+2+\cdots+n)+(n+1)) \\
&\quad \textit{(by induction hypothesis and using algebra)} \\
&= 1^3+2^3+\cdots+n^3+(n+1)\left(2\frac{n(n+1)}{2}+(n+1)\right) \\
&\quad \textit{(by the formula established in Problem 18.1)} \\
&= 1^3+2^3+\cdots+n^3+(n+1)^2(n+1) \\
&= 1^3+2^3+\cdots+n^3+(n+1)^3.
\end{aligned}$$

This concludes the induction step.

By the principle of mathematical induction, we conclude that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all $n \in \mathbb{Z}^+$.

Solution to Problem 18.6. *We first consider the base step $n = 0$. Since $1 + x > 0$, we can conclude that $(1 + x)^0 = 1 \geq 1 + 0x$. This establishes the base step.*

For the induction step we need to show that for $n \geq 0$, if $(1 + x)^n \geq 1 + nx$, then $(1 + x)^{n+1} \geq 1 + (n + 1)x$.

Now,

$$\begin{aligned}(1+x)^{n+1} &= (1+x)^n(1+x)\\ &\geq (1+nx)(1+x) \text{ (by induction hypothesis and the fact that } 1+x>0\text{)}\\ &= 1+nx+x+nx^2\\ &\geq 1+(n+1)x \text{ (since } nx^2\geq 0\text{)}.\end{aligned}$$

This concludes the induction step and Bernoulli's inequality is now established by the principle of mathematical induction.

Solution to Problem 18.9. *For the base step we let $n=0$. Then $5^{2n}-1=5^0-1=0$. Since 0 is divisible by any nonzero integer, the result holds in this case.*

For the induction step, let $n\geq 0$ and suppose that 8 divides $5^{2n}-1$. Then

$$\begin{aligned}5^{2(n+1)}-1 &= 5^2 5^{2n}-5^2+5^2-1\\ &= 5^2(5^{2n}-1)+5^2-1\\ &= 5^2(8k)+24 \text{ for some } k\in\mathbb{Z} \text{ (by the induction hypothesis)}\\ &= 8(5^2k+3), \text{ where } 5^2k+3\in\mathbb{Z}.\end{aligned}$$

This shows that $5^{2(n+1)}-1$ is divisible by 8 and concludes the induction step.

The result follows by mathematical induction.

Solution to Problem 18.12. *We will prove the inequality using induction.*
The base step $n=1$ is trivial: $|a_1-1|\leq|a_1-1|$.
Now let n be an integer with $n\geq 1$, and suppose that $\left|\left(\prod_{j=1}^n a_j\right)-1\right|\leq\sum_{j=1}^n|a_j-1|$. Hence

$$\begin{aligned}\left|\left(\prod_{j=1}^{n+1} a_j\right)-1\right| &= \left|\left(\prod_{j=1}^{n+1} a_j\right)-a_{n+1}+a_{n+1}-1\right|\\ &= \left|a_{n+1}\left(\left(\prod_{j=1}^{n} a_j\right)-1\right)+a_{n+1}-1\right|\\ &\leq |a_{n+1}|\left|\left(\prod_{j=1}^{n} a_j\right)-1\right|+|a_{n+1}-1| \text{ (by triangle inequality)}\\ &< \left|\left(\prod_{j=1}^{n} a_j\right)-1\right|+|a_{n+1}-1| \text{ (since } |a_{n+1}|<1\text{)}\\ &\leq \sum_{j=1}^{n}|a_j-1|+|a_{n+1}-1| \text{ (by induction hypothesis)}.\end{aligned}$$

Thus

$$\left| \left(\prod_{j=1}^{n+1} a_j \right) - 1 \right| \leq \sum_{j=1}^{n+1} |a_j - 1|.$$

This concludes the induction step.
The inequality now follows from the principle of mathematical induction.

Solution to Problem 18.15. *In the induction step of this "Not a proof," we reduce the polynomial p of positive degree to a polynomial q of positive degree that has one less linear factor than p but contains the factors ax and $a_1x + b_1$. This is not possible if p is of degree 2. Thus we make the implicit assumption that degree p is at least three. This means that the induction step is not applicable to show that if the statement holds for a degree 1 polynomial, then it also holds for a degree 2 polynomial. The induction fails at the step from P(1) to P(2). (The argument would be correct for all higher degrees, but this is irrelevant!)*

Solution to Problem 18.18. *We have:*

(a) $h(0) = \pi(g(0)) = \pi(1, 5) = 5$,
$h(1) = \pi(g(1)) = \pi(f(g(0))) = \pi(f(1, 5)) = \pi(2, \frac{5^2}{1}) = 5^2$, *and*
$h(2) = \pi(g(2)) = \pi(f(g(1))) = \pi(f(2, 5^2)) = \pi(3, 5^4/2) = 5^4/2$.

(b) *For* $n \in \mathbb{N}$ *we have* $h(n+1) = \pi(g(n+1)) = \pi(f(g(n))) = \pi(f(n+1, \pi(g(n)))) = \pi(f(n+1, h(n))) = \pi(n+2, h(n)^2/(n+1)) = h(n)^2/(n+1)$.
To summarize, h is defined recursively as

$$h(0) = 5; \quad \textit{and} \quad h(n+1) = \frac{h(n)^2}{n+1} \textit{ for } n \in \mathbb{Z}^+.$$

Solution to Problem 18.21. *The base step $n = 2$ holds since 2 is a prime number.*

Suppose that for some integer $n \geq 2$ and for all integers k, with $2 \leq k \leq n$, we know that k is prime or is the product of prime numbers.
If the integer $n + 1$ is prime, then we have established the induction step. If $n + 1$ is not prime, then $n + 1 = rs$, where r and s are integers satisfying $1 < r < n + 1$ and $1 < s < n + 1$. By the induction hypothesis, r is prime or the product of primes. Likewise, s is prime or the product of primes. Using these representations of r and s we can write $n + 1$ as a product of primes. This establishes the induction step for the remaining case.

By the second principle of mathematical induction we conclude that every integer n with $n \geq 2$ is a prime or a product of primes.

Solution to Problem 18.24. *(a) We claim that $T_n = n(n + 1)/2$ for $n \in \mathbb{Z}^+$.*

Proof. We use induction on n. For the base step $n = 1$, check that $T_1 = 1$.

For the induction step we let $n \geq 1$ and suppose that $T_n = n(n+1)/2$.

The triangular number T_{n+1} is obtained from the one before by extending the left and right side of the triangular array as shown in Figure 18.2 by one more line at the bottom. This line will necessarily have $n + 1$ new points. Hence

$$T_{n+1} = T_n + n + 1 = n(n+1)/2 + n + 1 = (n+1)(n+2)/2.$$

This establishes the induction step.

The principle of mathematical induction shows that the claim holds. □

(b) *We leave this one to you!*