Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 17: Images and Inverse Images

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Solution to Problem 17.3. *(a)* $f((-1,1)) = \{|x| : -1 < x < 1\} = [0,1)$;

(b) $f(\{-1,1\}) = \{|x| : x = -1 \text{ or } x = 1\} = \{1\}$;

(c) $f^{-1}(\{1\}) = \{x \in \mathbb{R} : |x| = 1\} = \{-1,1\}$;

(d) $f^{-1}([-1,0)) = \{x \in \mathbb{R} : -1 \leq |x| < 0\} = \emptyset$;

(e) $f^{-1}(f([0,1])) = f^{-1}(\{|x| : 0 \leq x \leq 1\}) = f^{-1}([0,1]) = \{x \in \mathbb{R} : 0 \leq |x| \leq 1\} = [-1,1]$.

Solution to Problem 17.6. *(a)* $\chi_{\mathbb{Z}}(\mathbb{Z}^+) = \{\chi_{\mathbb{Z}}(x) : x \in \mathbb{Z}^+\} = \{1\}$;

(b) $\chi_{\mathbb{Z}}^{-1}(\mathbb{Z}^+) = \{x \in \mathbb{R} : \chi_{\mathbb{Z}}(x) \in \mathbb{Z}^+\} = \{x \in \mathbb{R} : \chi_{\mathbb{Z}}(x) = 1\} = \mathbb{Z}$;

(c) $\chi_{\mathbb{Z}}(\chi_{\mathbb{Z}}^{-1}(\mathbb{Z}^+)) = \chi_{\mathbb{Z}}(\mathbb{Z}) = \{\chi_{\mathbb{Z}}(x) : x \in \mathbb{Z}\} = \{1\}$;

(d) $\chi_{\mathbb{Z}}^{-1}(\chi_{\mathbb{Z}}(\mathbb{Z}^+)) = \chi_{\mathbb{Z}}^{-1}(\{1\} = \{x \in \mathbb{R} : \chi_{\mathbb{Z}}(x) = 1\} = \mathbb{Z}$.

Solution to Problem 17.9. *We compute the answer using the definition of the image of a set. Now,*

$$\begin{aligned} f(2\mathbb{Z}) &= \{f(x) : x \in 2\mathbb{Z}\} \\ &= \{f(2m) : m \in \mathbb{Z}\} \\ &= \{f(2m) : m \in \mathbb{Z} \text{ and } m \leq 0\} \cup \{f(2m) : m \in \mathbb{Z} \text{ and } m > 0\} \\ &= \{-4m : m \in \mathbb{Z} \text{ and } m \leq 0\} \cup \{4m - 1 : m \in \mathbb{Z} \text{ and } m > 0\} \\ &= 4\mathbb{N} \cup (4\mathbb{Z}^+ - 1) \\ &= 4\mathbb{N} \cup (4\mathbb{N} + 3). \end{aligned}$$

Solution to Problem 17.12. *Proof.* If $z \in f(A_1 \cup A_2)$, then there is $x \in A_1 \cup A_2$ such that $z = f(x)$. If $x \in A_1$, then $z = f(x) \in f(A_1)$. If $x \notin A_1$, then $x \in A_2$. In this case $z = f(x) \in f(A_2)$. Thus in any case, $z = f(x) \in f(A_1) \cup f(A_2)$. This shows that $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$.

Conversely, if $z \in f(A_1) \cup f(A_2)$, then $z \in f(A_1)$ or $z \in f(A_2)$. If $z \in f(A_1)$, then there is $x \in A_1 \subseteq A_1 \cup A_2$ such that $z = f(x)$. Otherwise $z \in f(A_2)$ and again there is $x \in A_2 \subseteq A_1 \cup A_2$ such that $z = f(x)$. Thus, we conclude that $z \in f(A_1 \cup A_2)$. This shows that $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$.

Therefore $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$. $\square$

Solution to Problem 17.15. *Proof.* If $x \in f^{-1}(B_1 \cap B_2)$, then $x \in X$ and $f(x) \in B_1 \cap B_2$. Thus $x \in X$ and $f(x) \in B_1$. This shows that $x \in f^{-1}(B_1)$. We also have $x \in X$ and $f(x) \in B_2$. This shows that $x \in f^{-1}(B_2)$. We conclude that $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Hence $f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2)$.

Conversely, if $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$, then $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$. Thus $x \in X$, $f(x) \in B_1$, and $f(x) \in B_2$. This implies that $f(x) \in B_1 \cap B_2$. Hence $x \in f^{-1}(B_1 \cap B_2)$. We have shown that $f^{-1}(B_1) \cap f^{-1}(B_2) \subseteq f^{-1}(B_1 \cap B_2)$.

Thus $f^{-1}(B_1) \cap f^{-1}(B_2) = f^{-1}(B_1 \cap B_2)$. $\square$

Solution to Problem 17.18. *(a) Proof.* If $z \in f(f^{-1}(B))$, then there is $x \in f^{-1}(B)$ such that $z = f(x)$. Since $x \in f^{-1}(B)$, we conclude that $x \in X$ and $f(x) \in B$. Hence $z \in B$. This proves the set inclusion. $\square$

(b) We define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^x$. Then $f(f^{-1}(\mathbb{R})) = f(\mathbb{R}) = \mathbb{R}^+ \neq \mathbb{R}$.

(c) We claim that if the function $f : X \to Y$ is surjective, then $f(f^{-1}(B)) = B$.

Proof. If $b \in B$, then there is $x \in X$ with $f(x) = b$ because f is onto. Thus $x \in f^{-1}(B)$. This shows that $b = f(x) \in f(f^{-1}(B))$. Hence $B \subseteq f(f^{-1}(B))$. The reverse inclusion was proven in part (a). Therefore the two sets are equal. $\square$

(d) The example in part (b) shows that the two sets may not be equal even if the function $f : X \to Y$ is injective.

Solution to Problem 17.21. *Since partitions are only defined for nonempty sets, we may assume that $A \neq \emptyset$. This implies that $B \neq \emptyset$.*

Let $b \in B$. Since the function f is onto, there exists $a \in A$ such that $f(a) = b$. Thus $a \in f^{-1}(\{b\})$. This shows that $f^{-1}(\{b\}) \neq \emptyset$ for all $b \in B$.

If $a \in A$, then $f(a) = b \in B$. Thus $a \in f^{-1}(\{b\})$ for $b = f(a)$. Thus $a \in \bigcup_{b \in B} f^{-1}(\{b\})$. The reverse inclusion is trivial, thus $\bigcup_{b \in B} f^{-1}(\{b\}) = A$.

Let $f^{-1}(\{b\}) \cap f^{-1}(\{c\}) \neq \emptyset$ for some $b, c \in B$. Hence there exists $x \in f^{-1}(\{b\}) \cap f^{-1}(\{c\})$. Since $x \in f^{-1}(\{b\})$, we have $f(x) = b$. Also, $x \in f^{-1}(\{c\})$ and thus $f(x) = c$. We conclude that $b = c$. Hence $f^{-1}(\{b\}) = f^{-1}(\{c\})$.

We have thus shown that $\{f^{-1}(\{b\}) : b \in B\}$ partitions A.

Solution to Problem 17.24. *(a) We claim that $\chi_{A_1} = \chi_{A_2}$ implies $A_1 = A_2$. If $x \in A_1$, then $\chi_{A_1}(x) = 1$. Hence $\chi_{A_2}(x) = 1$. Thus $x \in A_2$. We have shown that $A_1 \subseteq A_2$. The reverse inclusion can be handled using the same argument, and the claim is then proven.*

(b) If $x \in X$, then $x \in A_1 \cap A_2$ or $x \notin A_1 \cap A_2$. In the first case, $x \in A_1$ and $x \in A_2$. Hence $\chi_{A_1}(x) = \chi_{A_2}(x) = 1$. In this case, $\chi_{A_1}(x) \cdot \chi_{A_2}(x) = 1 \cdot 1 = 1 = \chi_{A_1 \cap A_2}(x)$.

In the second case $x \notin A_1$ or $x \notin A_2$. Thus $\chi_{A_1}(x) = 0$ or $\chi_{A_2}(x) = 0$. This implies that $\chi_{A_1}(x) \cdot \chi_{A_2}(x) = 0 = \chi_{A_1 \cap A_2}(x)$.

We have shown that for all $x \in X$, we have $\chi_{A_1}(x) \cdot \chi_{A_2}(x) = \chi_{A_1 \cap A_2}(x)$. Hence $\chi_{A_1} \cdot \chi_{A_2} = \chi_{A_1 \cap A_2}$.

(c) We break the proof into four cases: 1) $x \notin A_1 \cup A_2$, 2) $x \in A_1 \setminus A_2$, 3) $x \in A_2 \setminus A_1$, or 4) $x \in A_1 \cap A_2$. Note that every element of X is in exactly one of the four cases. There are sets A_1 and A_2 for which some of the cases do not occur (the corresponding sets are empty).

It is now easy to check that in all four cases $\chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = \chi_{A_1 \cup A_2}(x)$. This proves the formula.

(d) Using the formula from part (c) and noticing that $(X \setminus A_1) \cup A_1 = X$, $(X \setminus A_1) \cap A_1 = \emptyset$, $\chi_X = 1$, and $\chi_\emptyset = 0$, it is straightforward to check that $\chi_{X \setminus A_1} = 1 - \chi_{A_1}$.