Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 16: Inverses

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If you discover errors in these solutions or feel you have a better solution, please write to us at udaepp@bucknell.edu or pgorkin@bucknell.edu. We hope that you have fun with these problems.

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Solution to Problem 16.3. *One way to do this is to use the result of Theorem 16.8 part (ii). In other words, if we show that f is not one-to-one, then no such g can exist. But $f(0,0) = f(1,-1)$ and $(0,0) \neq (1,-1)$ so f is not one-to-one and the result follows.*

Solution to Problem 16.6. *(a) A calculation shows that $(f \circ g)(x) = x$ for all $x \in \mathbb{R} \setminus \{1\}$. A second calculation shows that $(g \circ f)(x) = x$ for all $x \in \mathbb{R} \setminus \{-2\}$.*

(b) Theorem 16.8 implies that g and f are inverses of each other.

Solution to Problem 16.9. *This is essentially the same proof as the solution to Exercise 16.9, parts (i) and (ii) in the text. We leave it to you to adapt that proof to the particular situation addressed here.*

Solution to Problem 16.12. *We present the answers below. You should provide the details.*

(a) An example of such a function is $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = -x$. Theorem 16.8 implies that f must be one-to-one and onto.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ if x is an integer and $f(x) = \lfloor x \rfloor$ otherwise. Then we check the required condition in two stages:

If x is an integer, $(f \circ f)(x) = f(0) = 0$. If x is not an integer, $(f \circ f)(x) = f(\lfloor x \rfloor) = 0$. So $f \circ f = 0$ and f does what it needs to do.

Such a function cannot be onto and it cannot be one-to-one:

For onto: We know that f is not the zero function so there exists $y \in \mathbb{R}$ such that $f(y) \neq 0$. If f were onto, we would have an $x \in \mathbb{R}$ such that $f(x) = y$. But then $(f \circ f)(x) = f(y) \neq 0$, a contradiction.

For one-to-one: If f were one-to-one, then f would be a bijective mapping onto its range. Therefore, there would exist $g : \text{ran } f \to \mathbb{R}$ such that $(g \circ f) = i_{\mathbb{R}}$. But $g \circ (f \circ f) = (g \circ f) \circ f = f$ and therefore $g(0) = (g \circ f)(f(x)) = f(x)$ for all $x \in \mathbb{R}$. But this implies that f is the constant function $f(x) = g(0)$ for all x, which is impossible for a one-to-one function.

Solution to Problem 16.15. *This is indeed an equivalence relation!*
If $a \in A$, then $f(a) = f(a)$ and $a \sim a$. Thus $\sim$ is reflexive.
If $a, b \in A$ with $a \sim b$, then $f(a) = f(b)$. Therefore $f(b) = f(a)$ and $b \sim a$. Thus, $\sim$ is symmetric.
If $a, b, c \in A$ and $a \sim b$, $b \sim c$, then $f(a) = f(b)$ and $f(b) = f(c)$. Therefore $f(a) = f(c)$ and $a = c$. Thus, $\sim$ is transitive.
If f is one-to-one, then $a \sim b$ if and only $f(a) = f(b)$. This, in turn, happens if and only if $a = b$. Therefore, the equivalence class of a is $\{a\}$.

Solution to Problem 16.18. *If $A \cap C \neq \emptyset$ and $f \neq g$ on $A \cap C$, then H will not define a function.*

If $A \cap C = \emptyset$ and $x \in A \cup C$ with $H(x) = u_1$ and $H(x) = v_1$, then x is in A or $x \in C$ and we cannot have x in both sets. If $x \in A$, then $u_1 = f(x)$ and $u_2 = f(x)$. Since f is a function $u_1 = u_2$. Similarly, if $x \in C$ then $u_1 = g(x)$ and $u_2 = g(x)$. Since g is a function, $u_1 = u_2$. Therefore, in either case, $u_1 = v_1$. So H is a function if $A \cap C = \emptyset$.

A similar argument shows that if $A \cap C \neq \emptyset$ and $f = g$ on $A \cap C$, then H is a function.

Solution to Problem 16.21. *(a) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = |n|$. Then f does not map onto $\mathbb{Z}$. Define $g : \mathbb{Z} \to \mathbb{N}$ by $g(m) = |m|$. Then $g \circ f : \mathbb{Z} \to \mathbb{N}$ and $(g \circ f)(n) = |n|$ maps onto $\mathbb{N}$. Note that the g we choose must map onto C.*

(b) Let $f : \mathbb{Z}^+ \to \mathbb{Z}$ be defined by $f(n) = n$. Define $g : \mathbb{Z} \to \mathbb{N}$ by $g(m) = m^2$. Then g is not one-to-one. But $g \circ f : \mathbb{Z}^+ \to \mathbb{N}$ is $(g \circ f)(n) = n^2$ and this is one-to-one!