Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 15: Functions, One-to-One, and Onto

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Solution to Problem 15.3. *Since $f(3) = f(-3)$ and $3 \neq -3$, the function f is not one-to-one.*

Solution to Problem 15.6. *The definition says "For each $y \in Y$ there exists (at least one) $x \in X$ such that $f(x) = y$." The not-a-definition presented in this problem has changed the order of the quantifiers. Does it matter? In this case, it matters a lot. It says that the same x must work for all y. If this were the definition, there would be very few surjective functions! So the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(z) = z$ (which we surely would like to be surjective) would not qualify!.*

Which functions would qualify under this definition? (This question has a nice answer.)

Solution to Problem 15.9. *We are presenting the solution below, but our "devise a plan" step is not presented. In this case you should sketch the graph of this function to see how the natural division of the domain affects the proof of injectivity. The graph will also show you how the range of the graph helped us in the proof of surjectivity.*

We first prove that f is injective.
Let $a, b \in \mathbb{R}$ and suppose that $f(a) = f(b)$. There are several cases to consider.
Case 1: $a, b \leq -2$. Then $f(a) = a^2 + 2a + 4 = f(b) = b^2 + 2b + 4$. This simplifies to $a^2 - b^2 = -2(a - b)$, which implies that $(a + b)(a - b) = -2(a - b)$. If $a \neq b$, then $a + b = -2$ which contradicts the fact that $a \leq -2$ and $b \leq -2$. Hence we have $a = b$.
Case 2: $-2 < a, b < 2$. Then $f(a) = -2a = f(b) = -2b$, which implies that $a = b$.
Case 3: $a, b \geq 2$. Then $f(a) = -2 - a = f(b) = -2 - b$. This also implies that $a = b$.
Case 4: $a \leq -2$ and $-2 < b < 2$. Then $f(a) = a^2 + 2a + 4 = (a + 1)^2 + 3 \geq 4$ and $f(b) = -2b < 4$. Thus $f(a) = f(b)$ is not possible, this case cannot occur. (Note that this is the case "One of the two numbers is less than or equal to -2 and the other is strictly between -2 and 2.)

Case 5: $a \le -2$ *and* $b \ge 2$. *We again have* $f(a) \ge 4$. *Now* $f(b) = -2 - b \le -4$. *Thus* $f(a) = f(b)$ *is not possible, and this case cannot occur. (Again, this is the case "One of the two numbers is less than or equal to* -2 *and the other is at least* 2.*)*
Case 6: $-2 < a < 2$ *and* $b \ge 2$. *We have* $f(a) = -2a > -4$ *and* $g(b) = -2 - b \le -4$. *Thus, again,* $f(a) = f(b)$ *is not possible and this case cannot occur. (Note that there are no remaining cases to consider.)*
We conclude that whenever $f(a) = f(b)$, *then* $a = b$. *Thus, this function is injective.*

Now we will prove that f *is surjective. Let* $z \in \mathbb{R}$ *and we will consider three cases.*
Case 1: $z \ge 4$. *We set* $x = -\sqrt{z-3} - 1$. *Note that* $z - 3 \ge 1$ *and thus* $x \in \mathbb{R}$ *and* $x \le -2$. *Thus* $f(x) = (-\sqrt{z-3} - 1)^2 + 2(-\sqrt{z-3} - 1) + 4 = z - 3 + 2\sqrt{z-3} + 1 - 2\sqrt{z-3} - 2 + 4 = z$. *Hence* $z \in ran(f)$.
Case 2: $-4 < z < 4$. *We set* $x = -z/2$. *Then* $x \in \mathbb{R}$ *and* $-2 < x < 2$. *Hence* $f(x) = -2(-z/2) = z$. *Thus,* $z \in ran(f)$.
Case 3: $z \le -4$. *We set* $x = -2 - z$. *Then* $x \in \mathbb{R}$ *and* $x \ge 2$. *Hence* $f(x) = -2 - (-2 - z) = z$. *Thus* $z \in ran(f)$.
We conclude that $\mathbb{R} \subseteq ran(f)$. *Since the reverse inclusion is obvious, we have shown that the function is surjective. Together with the first part of the proof this shows that* f *is a bijection.*

Solution to Problem 15.12. *We present the answers below. You should provide the details.*

(a) *This is a function, it is one-to-one and it is onto.*

(b) *This is a function. It is not one-to-one and it is not onto. (What would have to be mapped to* 3*?)*

(c) *This is always a function. If* $y = 0$ *this is neither one-to-one nor onto. If* $y \ne 0$, *then it is both one-to-one and onto.*

(d) *This is not a function. When* $S = \emptyset$, *the maximum is not defined. What if* $S \ne \emptyset$*?*

Solution to Problem 15.15. *We note that* ϕ *is a function.*
This is not one-to-one because two different functions can map zero to the same value; that is, if we take f *defined by* $f(x) = 0$ *for all* x *and we take* g *defined by* $g(x) = x$ *for all* x, *then* $f \ne g$ *but* $\phi(f) = f(0) = 0 = g(0) = \phi(g)$.
This is onto, however. Take $x \in \mathbb{R}$ *and note that the constant function* f_x *defined by* $f_x(y) = x$ *for all* $y \in [0,1]$ *is an element of* $F([0,1])$. *Therefore,* $\phi(f_x) = f_x(0) = x$. *So* ϕ *is onto.*

Solution to Problem 15.18. *By definition,* $f|_C : C \to B$. *Hence a domain and codomain are specified. Now let* $x \in C$. *Since* $C \subseteq A$ *we also have* $x \in A$. *Hence there is* $y \in B$ *with* $f(x) = y$. *By the definition of restrictions,* $f|_C(x) = f(x) = y$.

For the second requirement, suppose that for $x \in C$ *we have* $y, z \in B$ *such that* $f|_C(x) = y$ *and* $f|_C(x) = z$. *By the definition of restriction, we get* $y = f|_C(x) = f(x) = f|_C(x) = z$. *Hence* $y = z$.

This concludes the proof that the restriction of a function is a function.

Solution to Problem 15.21. *Clearly f is not a bijection (e.g. $f(0) = f(\sqrt{3}) = 0$). We claim that $f|_A : A \to \mathbb{R}$ is a bijection.*

Proof. We will first prove that $f|_A : A \to \mathbb{R}$ is injective. Suppose that for $x_1, x_2 \in A$ and $x_1 < x_2$ we have $f|_A(x_1) = f|_A(x_2)$. Then

$$\begin{aligned} x_1^3 - 3x_1 &= x_2^3 - 3x_2 \\ (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) &= 3(x_1 - x_2) \\ x_1^2 + x_1x_2 + x_2^2 &= 3, \quad \text{since } x_1 \neq x_2 \end{aligned}$$

We consider three cases. In the first case, $x_1 < x_2 < -\sqrt{3}$. Then $x_1^2 + x_1x_2 + x_2^2 > 9$. This contradicts the last line above. In the second case $\sqrt{3} \leq x_1 < x_2$ and we also conclude that $x_1^2 + x_1x_2 + x_2^2 > 9$, reaching the same contradiction. In the last case, $x_1 < -\sqrt{3} < \sqrt{3} \leq x_2$. Then $f|_A(x_1) = x_1(x_1^2 - 3) < 0$ and $f|_A(x_2) = x_2(x_2^2 - 3) \geq 0$. We again reach a contradiction. This shows that the function is injective.

We will now show that the function $f|_A$ is surjective using techniques from calculus. Let $y \in \mathbb{R}$ and first suppose that $y \geq 0$. Let $a = \max\{3, y\}$. Then $a \geq y$ and $a \geq 3$. Thus $a^2 \geq 9$. Hence $a^2 - 3 \geq 6 > 1$. We conclude that $f(a) = a(a^2 - 3) > a \geq y$. Since f is a polynomial, the function $f|_A$ is continuous on $[\sqrt{3}, a] \subseteq A$ and $0 = f|_A(\sqrt{3}) \leq y \leq f|_A(a)$. By the intermediate value theorem, there is a real number c with $c \in [\sqrt{3}, a] \subseteq A$ such that $f|_A(c) = y$.

For the case of $y < 0$ we use an analogous argument: Let $b = \min\{-3, y\}$. Then $b \leq y$ and $b \leq -3$. Thus $b^2 \geq 9$. Hence $b^2 - 3 \geq 6 > 1$. We conclude that $f(b) = b(b^2 - 3) < b \leq y$. Since f is a polynomial, it is continuous on $[b, -\sqrt{3}]$ and $f(b) < y < 0 = f(-\sqrt{3})$. By the intermediate value theorem, there is a real number d with $d \in (b, -\sqrt{3})$ such that $f(d) = y$. Since $d < -\sqrt{3}$ we have $d \in A$ and $f|_A(d) = f(d) = y$. This shows that $f|_A$ is also surjective and hence bijective. □

Solution to Problem 15.24. *(a) We claim that the function f is one-to-one if and only if $ad - bc \neq 0$.*

We first assume that $ad - bc \neq 0$. If $x, y \in dom(f)$ and $f(x) = f(y)$, then

$$\begin{aligned} \frac{ax+b}{cx+d} &= \frac{ay+b}{cy+d}, \\ (ax+b)(cy+d) &= (ay+b)(cx+d), \\ acxy + adx + bcy + bd &= acxy + ady + bcx + bd, \\ ad(x-y) - bc(x-y) &= 0, \\ (x-y)(ad-bc) &= 0. \end{aligned}$$

Since $ad - bc \neq 0$ we conclude that $x - y = 0$ and thus $x = y$. This shows that f is one-to-one.

For the converse we assume that $ad - bc = 0$. Since not both of c and d are zero, we first assume that

$c \neq 0$. *Then* $b = ad/c$. *Note also that* $cx + d \neq 0$ *for* $x \in dom(f)$. *We calculate*

$$\begin{aligned} f(x) &= \frac{ax+b}{cx+d} \\ &= \frac{ax+\frac{ad}{c}}{cx+d} \\ &= \frac{a}{c}\frac{cx+d}{cx+d} \\ &= \frac{a}{c}. \end{aligned}$$

Thus f *is the constant function on* $X = \mathbb{R} \setminus \{-d/c\}$. *Since this domain has at least two elements,* f *is not one-to-one.*

We now consider the remaining case of $c = 0$. *Then* $d \neq 0$. *Since* $ad - bc = 0$, *we get* $ad = 0$. *Thus we conclude that* $a = 0$. *This simplifies the function to* $f(x) = \frac{b}{d}$. *In this case we have* $X = \mathbb{R}$. *Again we have a constant function on a domain with at least two elements. This shows that the function is not one-to-one in this case either.*

The two parts together establish the claim.

(b) *We now assume that* $c = 0$ *and* $ad - bc \neq 0$. *This implies that* $ad \neq 0$. *Thus* $a \neq 0$ *and* $d \neq 0$. *Hence* $X = \mathbb{R}$. *The function will then simplify to* $f(x) = \frac{a}{d}x + \frac{b}{d}$. *We already know from part (a) that this function is one-to-one. It only remains to show that it is onto.*

For this purpose, let $z \in \mathbb{R}$. *We set* $x = (dz - b)/a$. *Since* $a \neq 0$, *we have* $x \in \mathbb{R} = dom(f)$. *Now* $f(x) = \frac{a}{d}\frac{dz-b}{a} + \frac{b}{d} = \frac{dz-b+b}{d} = z$. *This shows that the function is also onto. Thus* f *is a bijection from* $\mathbb{R}$ *to* $\mathbb{R}$.