Reading, Writing, and Proving (Second Edition)

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Springer Verlag, 2011

Solutions to Chapter 14: Functions, Domain, and Range

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Solution to Problem 14.3. (a) *This is not a function, since* $(0,2),(0,-2) \in f$ *violating condition (ii) of the function definition.*

(b) *This is not a function, no value in* $\mathbb{R}$ *is related to* -1 *of the domain. Thus condition (i) of the function definition is violated.*

(c) *This is a function. The domain and codomain are specified. For every* $(x,y) \in dom(f)$*, we have* $f(x,y) = x+y \in \mathbb{R}$*. Thus condition (i) of the definition holds. Suppose that for* $(x,y) \in \mathbb{R}^2$ *we have* $f(x,y) = a$ *and* $f(x,y) = b$*. Then* $a = f(x,y) = x+y$ *and* $b = f(x,y) = x+y$*. Thus* $a = b$ *and condition (ii) of the definition also holds.*

(d) *This is a function. The domain and codomain are specified and each interval of the form* $[a,b]$ *with* $a \leq b$ *has precisely one minimum (precisely one left-hand endpoint). Therefore, if* $f([a,b]) = a_1$ *and* $f([a,b]) = a_2$*, then* $a_1 = a_2 = a$*.*

(e) *This is a function. The domain and codomain are specified. For each* $(n,m) \in \mathbb{N} \times \mathbb{N} = dom(f)$ *we have* $f(n,m) = m \in \mathbb{R}$*, showing that condition (i) holds. Suppose that for* $(n,m) \in dom(f)$ *we have* $f(n,m) = a$ *and* $f(n,m) = b$*. Then* $a = f(n,m) = m$ *and* $b = f(n,m) = m$ *and thus* $a = b$*, showing that condition (ii) also holds.*

(f) *This is a function. The domain and codomain are both defined and for any* $(n,m) \in \mathbb{N} \times \mathbb{N} = dom(f)$*, the value* $f(n,m) = m \in cod(f)$*. Suppose that* $f(n,m) = a$ *and* $f(n,m) = b$*. Then* $a = f(n,m) = m$ *and* $b = f(n,m) = m$*, thus* $a = b$*. This shows that this is a well-defined function.*

(g) *This is a function. The domain and codomain are both specified. For* $x \in \mathbb{R} = dom(f)$*, we have* $f(x) = 0 \in cod(f)$ *if* $x \geq 0$*; or* $f(x) = x \in cod(f)$ *if* $x \leq 0$*. In either case,* $f(x) \in cod(f)$ *is defined. Now suppose that for some* $x \in \mathbb{R}$ *we have* $a = f(x)$ *and* $b = f(x)$*. If* $x > 0$*, then* $a = f(x) = 0$ *and* $b = f(x) = 0$*. Thus* $a = b$*, in this case. If* $x < 0$*, then* $a = f(x) = x$ *and* $b = f(x) = x$*. Thus* $a = b$ *in this case as well. If* $x = 0$*, then* $a = f(x) = 0$ *and* $b = f(x) = 0$*, thus* $a = b$*. Note that in the last case either rule of the function leads to* $f(x) = 0$*. Hence for all* $x \in dom(f)$*, if* $a = f(x)$ *and* $b = f(x)$*, then* $a = b$*. This concludes the proof that* f *as given is a function.*

(h) *This is not a function: We note that* $0 \in 2\mathbb{Z}$ *and* $0 \in 3\mathbb{Z}$ *and that* $f(0) = 1$ *and* $f(0) = -1$*, violating condition (ii) of the definition of a function.*

(i) *This is a function. The domain and codomain are both specified. Given any circle* $c \in dom(f)$*, the circumference of* c *is a real number and thus* $f(c) \in cod(f)$*. If for some* $c \in dom(f)$*,* $a = f(c)$ *and* $b = f(c)$*, then* $a = f(c) =$ *circumference of* c *and* $b = f(c) =$ *circumference of* c*. Since the circumference of a circle is a unique real number, we conclude that* $a = b$*. This concludes the proof that* f *is a function.*

(j) *This is a function. The domain and codomain are specified. If* $p \in dom(f)$*, then* $p(x) = a_n x^n + \ldots + a_1 x + a_0$*, where* a_k *are real numbers. Then* $f(p) = p'$ *and* $p'(x) = n a_n x^{n-1} + \ldots + a_1$ *is again a polynomial with real coefficients. Thus* $f(p) \in cod(f)$*. Suppose now that for some polynomial* p*, we have* $q = f(p)$ *and* $r = f(p)$*. Then* $q = f(p) = p'$ *and* $r = f(p) = p'$*. Since the derivative of a differentiable function (in particular of a polynomial) is unique, we conclude that* $q = r$*. Thus* f *as defined is a function.*

(k) *This is a function. The domain and codomain are both specified. Let* $p \in dom(f)$*. Then* $p(x) = a_n x^n + \ldots + a_1 x + a_0$*, where* a_k *are real numbers. Thus*

$$\begin{aligned} f(p) &= \int_0^1 p(x)dx \\ &= \left(\frac{a_n}{n+1}x^{n+1} + \ldots + \frac{a_1}{2}x^2 + a_0 x\right)\Bigg|_0^1 \\ &= \frac{a_n}{n+1} + \ldots + \frac{a_1}{2} + a_0 \in \mathbb{R} = cod(f). \end{aligned}$$

Suppose now that for some polynomial p*, we have* $a = f(p)$ *and* $b = f(p)$*. Then* $a = f(p) = \int_0^1 p(x)dx$ *and* $b = f(p) = \int_0^1 p(x)dx$*. Since the definite integral of an integrable function (in particular of a polynomial) is unique, we conclude that* $a = b$*. Thus* f *as defined is a function.*

Solution to Problem 14.6. (a) *A domain and codomain are specified. For* $x \in dom(\chi_A)$ *we have* $\chi_A(x) = 0 \in cod(\chi_A)$ *or* $\chi_A(x) = 1 \in cod(\chi_A)$*. So in both cases* $\chi_A(x)$ *is an element in* $cod(\chi_A)$*. Suppose that for some* $x \in X = dom(\chi_A)$ *we have* $\chi_A(x) = a$ *and* $\chi_A(x) = b$*. If* $x \in A$*, then* $a = \chi_A(x) = 1$ *and* $b = \chi_A(x) = 1$*, thus* $a = b$*. If* $x \notin A$*, then* $a = \chi_A(x) = 0$ *and* $b = \chi_A(x) = 0$*, thus* $a = b$*. In both cases,* $a = b$ *and hence the characteristic function is a function.*

(b) *As given,* $dom(\chi_A) = X$*. There are three possibilities for the range of* χ_A*:*

$$ran(\chi_A) = \begin{cases} \{0\} & \text{if } A = \emptyset \\ \{1\} & \text{if } A = X \\ \{0,1\} & \text{if } \emptyset \subset A \subset X. \end{cases}$$

Solution to Problem 14.9. *(a) We can describe the function by defining f on $\mathbb{R}$ by*

$$f(x) = (2-x)\chi_{(-\infty,0]}(x) + 2\chi_{(0,2)}(x) + x\chi_{[2,\infty)}(x)$$

This is not a step function because the range takes on infinitely many values.

(b) We can describe the function by defining g on $[0,12]$ by

$$g(x) = \chi_{[2,5)}(x) + \frac{1}{2}\chi_{\{5\}}(x) - \chi_{(5,10]}(x).$$

This is a step function, although to satisfy the definition we would need to write it as

$$g(x) = 0\chi_{[0,2)}(x) + 1\chi_{[2,5)}(x) + \frac{1}{2}\chi_{\{5\}}(x) + (-1)\chi_{(5,10]}(x) + 0\chi_{(10,12]}(x).$$

(c) We can write this function as

$$h(x) = \sin(x)\chi_{[0,2\pi]}(x),$$

where $x \in \mathbb{R}$. This is not a step function, because the range contains infinitely many (distinct) values.

Solution to Problem 14.12. *(a) One way to do this is*

$$\lfloor x \rfloor = \sum_{k=-\infty}^{\infty} k\chi_{[k,k+1)}(x).$$

(We note that the fact that this sum is infinite is problematic and requires further discussion. This can, however, be justified rigorously and you will do so in an analysis class. In fact, for each value x you only need exactly one term of this horrendous sum!)

The following is an alternate solution:

$$\lfloor x \rfloor = \sum_{k=1}^{\infty} \chi_{[k,\infty)}(x) - \sum_{k=0}^{\infty} \chi_{(-\infty,-k)}(x).$$

(b) It is not a step function since the range has infinitely many (distinct) values.

Solution to Problem 14.15. *We claim that $ran(f) = \mathbb{R} \setminus \{0\}$.*

If $z \in ran(f)$, then there is $x \in dom(f)$ with $z = f(x) = 1/(x-2)$. Since $x \neq 2$ we conclude that $z \in \mathbb{R}$. Suppose now that $z = 0$, then $\frac{1}{x-2} = 0$, which implies that $1 = 0$. Since this is impossible we cannot have $z = 0$. We conclude that $z \in \mathbb{R} \setminus \{0\}$; that is, $ran(f) \subseteq \mathbb{R} \setminus \{0\}$.

If $z \in \mathbb{R} \setminus \{0\}$, we set $x = \frac{1+2z}{z}$. Since $z \neq 0$, we have $x \in \mathbb{R}$. Suppose that $x = 2$. Then $\frac{1+2z}{z} = 2$, which implies that $1 + 2z = 2z$. Thus we would conclude that $1 = 0$. This contradiction shows that $x \neq 2$. Hence $x \in dom(f)$. Now $f(x) = \frac{1}{x-2} = \frac{1}{\frac{1+2z}{z} - 2} = \frac{z}{1+2z-2z} = z$. So $z \in ran(f)$ and $\mathbb{R} \setminus \{0\} \subseteq ran(f)$.

Together, these two parts prove that $ran(f) = \mathbb{R} \setminus \{0\}$.

Solution to Problem 14.18. *You should provide the details for each of the examples below. There are many different possible answers, so you should also try to think of some other examples.*

(a) *Let* $f(x, y) = |x| + 1$.

(b) *Let* $f(x, y) = |x| + |y|$. *While the range of* f *is clearly a subset of* $\mathbb{N}$, *don't forget to show that it is actually equal to* $\mathbb{N}$.

(c) *Let* $f(x, y) = 1$ *for all* $(x, y) \in \mathbb{Z} \times \mathbb{Z}$.

Solution to Problem 14.21. *No, this does not necessarily define a function. For example, if* $f : \mathbb{Z} \to \mathbb{N}$ *is defined by* $f(x) = x^2$, *then the set we consider is* $S = \{(x^2, x) : x \in \mathbb{Z}\}$. *But this is not a function for two reasons. Here is one of them: Let* $x_1 = 1$ *and* $x_2 = -1$. *Then* $x_1 \neq x_2$, *but* $(1, 1) \in S$ *and* $(1, -1) \in S$ *and* $1 \neq -1$. *So this does not satisfy the third condition in Definition 14.1.*

To answer this question, we are required to give only one reason that this is not a function. But this is not a function on two counts. What's the second reason that this is not a function?

Solution to Problem 14.24. *We provide the description of the set. A complete answer would include justification for the set equalities below.*

(a) $\bigcup_{n \in \mathbb{Z}^+} \{x \in \mathbb{R} : \pi - 2n \leq x \leq \pi + 2/n\} = (-\infty, 2 + \pi]$.

(b) $\bigcap_{n \in \mathbb{Z}^+} \{x \in \mathbb{R} : \pi - 2n \leq x \leq \pi + 2/n\} = [\pi - 2, \pi]$.