Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 13: Consequences of the completeness of $\mathbb{R}$

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If you discover errors in these solutions or feel you have a better solution, please write to us at udaepp@bucknell.edu or pgorkin@bucknell.edu. We hope that you have fun with these problems.

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Solution to Problem 13.3. *Let $T_1 = [1/2, 3/4]$. Then $T_1 \subset S$. Now $1/2 \leq t$ for all $t \in T_1$ and $1/2 \in T_1$. Therefore, if u is any lower bound of T_1, then $u \leq 1/2$. Thus $1/2$ is the infimum. Similarly, $3/4$ is the supremum. Since $1/2, 3/4 \in S$, we have completed the first part of the problem.*
Now consider the interval $T_2 = (0, 1)$. Then $0 \notin T_2$ and $1 \notin T_2$. We claim that $0 = \inf T_2$. Clearly, 0 is a lower bound. If m is any other lower bound, then $m < 1$. Suppose m is also a lower bound and $m > 0$. Then $m/2 < m$ and $m/2 \in T_2$. But then m is not a lower bound. So every lower bound $m \leq 0$, and thus 0 is the greatest lower bound. Now 1 is an upper bound. If $M < 1$, and M is an upper bound, then $M > 0$ and $0 < M < (M+1)/2 < 1$. But then $(M+1)/2 \in (0,1)$ and therefore M is not an upper bound, a contradiction. Thus 1 is the supremum and $1 \notin T_2$.

Solution to Problem 13.6. *We let $S = \{x, y\}$.*

(a) *We claim*

$$\frac{|x-y|+x+y}{2} = \frac{x-y+x+y}{2} = \max\{x, y\}.$$

To see this, we study the three possible cases. First note that if $x = y$, then our formula produces the correct answer. Second, if $x > y$, then we get

$$\frac{|x-y|+x+y}{2} = \frac{(x-y)+x+y}{2} = x.$$

Finally, if $y > x$ then

$$\frac{|x-y|+x+y}{2} = \frac{(y-x)+x+y}{2} = y.$$

(b) *In much the same way as above, it can be shown that*

$$\min\{x, y\} = -\frac{|x-y|-x-y}{2} = \frac{x+y-|x-y|}{2}.$$

(c) For a two element set (or any finite set) the infimum (supremum) and the minimum (maximum) are the same.

Solution to Problem 13.9. *(a) This is true: Since S is finite, S has a maximum and minimum (we proved this earlier, using the well-ordering principle for $\mathbb{N}$). We also showed that these are the supremum and infimum, respectively. Since the maximum and minimum must belong to the set, the result follows.*

(b) This is true: Clearly, $T \subseteq S$. Suppose that $s \in S$. Since U is the supremum of S, we know that $s \leq U$. Therefore $s \in T$ and we have $S \subseteq T$.

(c) This is false: Consider $S = [0, 2]$. Then we know that $U = 2$, since U is the maximum of S. Now $v = \sup\{x \in S : x < 2\}$, which is just the supremum of the interval $[0, 2)$. You should provide the details, if you haven't already done so, that $v = 2$. Therefore $v = U$, and the result is false.

Solution to Problem 13.12. *Let a and b be as given. Let $a_1 = a/\sqrt{2}$ and $b_1 = b/\sqrt{2}$. Then $a_1 < b_1$. Using the density of the rationals (Theorem 12.12), choose a rational number r with $a_1 < r < b_1$. Then $a < r\sqrt{2} < b$. Since $r\sqrt{2} \notin \mathbb{Q}$, we have completed the proof.*

Solution to Problem 13.15. *Property (i) is violated: We have $0 \in \mathbb{R}$ but $0 \not< 0$.*
Property (iv) is violated: If $x = 0$ and $y = 0$, elements of $\mathbb{R}$ then we have $x \not< y$ and $y \not< x$.
Note that properties (ii) and (iii) hold.

Solution to Problem 13.18. *(a) There is an "upper set." Let $U = \{1, 2, 5, 7, 8, 10\}$. Then $X \subseteq U$ for all X in $\mathcal{B}$. This set is also a least upper set: If V is an upper set, then we must have $\{1, 2, 5, 7\} \subseteq V$, $\{2, 8, 10\} \subseteq V$, and $\{2, 5, 8\} \subseteq V$. Therefore, $U \subseteq V$ and U is a least upper set for $\mathcal{A}$.*

(b) Since every set in $\mathcal{P}(\mathbb{Z})$ is a subset of $\mathbb{Z}$, we see that $\mathbb{Z}$ serves as an upper set.

(c) A set $V \in \mathcal{P}(\mathbb{Z})$ is a lower set for $\mathcal{A}$ if $V \subseteq X$ for all $X \in \mathcal{A}$. We say that a nonempty subset $\mathcal{A}$ is lower bounded if there is a lower set for $\mathcal{A}$ in $\mathcal{P}(\mathbb{Z})$. Finally, $V_0 \in \mathcal{P}(\mathbb{Z})$ is a greatest lower set if it is a lower set of $\mathcal{A}$ and if V is another lower set of $\mathcal{A}$, then $V \subseteq V_0$.

(d) The least upper set of $\mathcal{A}$ can be written as $\bigcup_{X \in \mathcal{A}} X$. The greatest lower set of $\mathcal{A}$ can be written as $\bigcap_{X \in \mathcal{A}} X$. You should provide details for these claims.

(e) By definition, every upper bounded set in $\mathcal{P}(\mathbb{Z})$ is assumed nonempty. The previous part of this problem gives an expression for the least upper set of an arbitrary nonempty subset of $\mathcal{P}(\mathbb{Z})$.

Solution to Problem 13.21. *Let* $A = \{m \in \mathbb{Z}^+ : m\sqrt{a} > 10000\}$. *Note that* $A \subseteq \mathbb{N}$. *By the Archimedean property of* $\mathbb{R}$, *the set* A *is nonempty. Hence by the well-ordering principle of the natural numbers, the set* A *has a minimum that we will denote by* n. *Then* $n\sqrt{a} > 10000$ *and we have* $\sqrt{a} > \frac{10000}{n}$. *Since* $\sqrt{a} < 10000$ *we have* $n > 1$ *and, consequently,* $n - 1 > 0$. *Since* $n - 1 < min(A)$, *we see that* $n - 1 \notin A$. *Thus* $(n-1)\sqrt{a} \leq 10000$.

We claim that $(n-1)\sqrt{a}$ *is irrational. Suppose to the contrary that* $(n-1)\sqrt{a} = p/q$ *for some integers* p *and* q *with* $q \neq 0$. *Then* $a = p^2/((n-1)q)^2$ *with* $p^2, ((n-1)q)^2$ *integers and* $((n-1)q)^2 \neq 0$. *This would imply that* a *is rational, which is a contradiction. Hence the claim is established and we conclude that* $(n-1)\sqrt{a} < 10000$.

Thus $\sqrt{a} < \frac{10000}{n-1}$. *Taken together, the two inequalities show that there is a positive integer* n *such that*

$$\frac{10000}{n} < \sqrt{a} < \frac{10000}{n-1}.$$