Reading, Writing, and Proving (Second Edition)

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Springer Verlag, 2011

Solutions to Chapter 12: Order in the Reals

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If you discover errors in these solutions or feel you have a better solution, please write to us at udaepp@bucknell.edu or pgorkin@bucknell.edu. We hope that you have fun with these problems.

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Solution to Problem 12.3. *(a)* *Let* $E = (1, 4)$*. If* $x \in (1, 4)$*, then* $x < 4$*. Therefore* 4 *is an upper bound. To see that* 4 *is the least upper bound, suppose that we have an upper bound* r *of* E *that is less than* 4*. Clearly* $1 < r$*. Let* $s = (4 + r)/2$*. Since* s *is the average of* r *and* 4*, we see that* $1 < s < 4$ *and* $r < s$*. Therefore* $s \in (1, 4)$ *and* $s > r$*, so* r *is not an upper bound. Thus,* 4 *is the least upper bound.*

(b) *Since* $1.01 \in (1, 4)$ *and* $1.01 < 1.1$*, we see that* 1.1 *is not a lower bound.*

Solution to Problem 12.6. *Let* $S = \{1 - 1/n : n \in \mathbb{Z}^+\}$*. First, we note that* $1 - 1/n \leq 1$ *for every* $n \in \mathbb{Z}^+$ *and therefore* 1 *is an upper bound.*

We check that 1 *is the least upper bound of* S*. Note that if* r *is an upper bound of the set with* $r < 1$*, then* $1 - r > 0$*. Therefore, there exists a positive integer* m *with* $1 - r > 1/m$*. So* $1 - 1/m > r$*. Since* $1 - 1/m$ *is in our set (this is very important!) and* $1 - 1/m > r$*, we see that* r *cannot be an upper bound. Therefore* 1 *is the least upper bound.*

If you don't know why it is essential to check that $1 - 1/m$ *is in the set* S*, we suggest you work Problem 12.1.*

Solution to Problem 12.9. *We have assumed that* $S \neq \emptyset$*, so let* $s \in S$*. Then* $\inf S \leq s \leq \sup S$*, as desired. If* $S = \{s\}$*, then* $\inf S = s = \sup S$*. (Since* S *is a finite set, the infimum is the minimum and the supremum is the maximum of the set.) If* S *is a nonempty subset of* $\mathbb{R}$ *with more than one element, however, it is not possible (as you should check) for* $\inf S = \sup S$*.*

Solution to Problem 12.12. *(a)* *Let* U *be an upper bound of* S*. Then* $x + U \in \mathbb{R}$ *and* $x + s \leq x + U$ *for all* $s \in S$*. Therefore,* $x + U$ *is an upper bound of* $x + S$ *and the set is bounded above.*

(b) *If we take* U *to be the least upper bound of* S *it is, in particular, an upper bound of* S*. From our work above, we see that* $x + U$ *is an upper bound of* $x + S$*. Since* $\sup(x + S)$ *is the least upper bound of* $x + S$*, we conclude that* $\sup(x + S) \leq x + U = x + \sup S$*.*

(c) *Again we take $U = \sup S$. Let $v < x + U$. We must show that v is not an upper bound for $x + S$. Consider $v - x$. Then U is the least upper bound of S and $v - x < U$, so we see that $v - x$ cannot be an upper bound of S! Therefore, there exists $s \in S$ such that $v - x < s$. Consequently, $v < x + s$ and v is not an upper bound of $x + S$. This implies that there is no upper bound of $x + S$ smaller than $x + U$, so $\sup(x + S) \geq x + \sup S$.*

Using this and part (b) of this problem, we have $x + U = x + \sup S$ is the least upper bound of $x + S$; that is, $x + \sup S = \sup(x + S)$.

Solution to Problem 12.15. *First we show that 2 is an upper bound. Let $x \in (0, 2)$. Then $0 < x < 2$, so 2 is clearly an upper bound. Suppose to the contrary that 2 is not the supremum. Then there exists an upper bound u with $u < 2$. Since $1 \in (0, 1) \cap \mathbb{Q}$ we have $1 \leq u$. By Theorem 12.12, there is a rational number a with $u < a < 2$. Then $1 \leq u < 2$ and $a \in (0, 2) \cap \mathbb{Q}$.*

Hence we have shown that u is not an upper bound of $(0, 2) \cap \mathbb{Q}$ and we conclude that 2 must be the supremum.

A very similar proof shows that 0 is the infimum of the set $(0, 2) \cap \mathbb{Q}$.

Solution to Problem 12.18. *By the well-ordering principle of the natural numbers, we know that every nonempty subset of the natural numbers has a minimum. Let E be a nonempty bounded subset of the natural numbers and let M be an integer bound on E. Therefore, the set $F = \{(M - x : x \in E\}$ is also a nonempty subset of the natural numbers. By the well-ordering principle, F has a minimum, which we denote by m. Therefore, $m \in F$ and $M - x \geq m$ for all $x \in E$. Note that $m = M - x_0$ for some $x_0 \in E$. We claim that x_0 is the desired maximum.*

Since x_0 is in E, we need only show that it is greater than or equal to every element in E. So let $y \in E$. Then $M - y \in F$ and consequently $M - x_0 = m \leq M - y$. Therefore, $-x_0 \leq -y$ or $x_0 \geq y$, as desired.

Solution to Problem 12.21. *Let $M \in \mathbb{Z}^+$ be chosen with $M > |a|$. (Note that we have proved, in the corollary to the Archimedean theorem, the existence of such an integer.) Consider $a' = a + M$ and $b' = b + M$. Then $0 < a' < b'$. By the theorem, we know that there exists $r \in \mathbb{Q}$ with $a' < r < b'$. Therefore $a < r - M < b$ and $r - M$ is rational.*