Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 11: Partitions

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Solution to Problem 11.3. *(a) If we take $r = -1$, the set $\{x \in \mathbb{R} : |x| = -1\} = \emptyset$. Thus condition (i) of the definition of partition is violated.*

(b) There is no natural number n such that $n < |0|$. Thus 0 is not in any of the subsets of $\mathbb{R}$ specified by the collection and condition (ii) of the definition of partition is violated.

(c) We have $(\mathbb{R} \setminus \mathbb{R}^+) \cap (\mathbb{R} \setminus \mathbb{R}^-) = \{0\} \neq \emptyset$ but $\mathbb{R} \setminus \mathbb{R}^+ \neq \mathbb{R} \setminus \mathbb{R}^-$. Thus, condition (iii) of the definition of partition is violated.

Solution to Problem 11.6. *(a) For $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^2$, the equivalence relation would be $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if and only if $z_1 = z_2$. You should be able to explain why this is an equivalence relation and why it partitions $\mathbb{R}^3$ into horizontal planes.*

(b) The equivalence relation would be $(x_1, y_1, z_1) \sim_1 (x_2, y_2, z_2)$ if and only if $x_1^2 + y_1^2 + z_1^2 = x_2^2 + y_2^2 + z_2^2$. Note that we think of the point $(0, 0, 0)$ as a sphere of radius 0. (It's a very little sphere.) You should be able to explain why this is an equivalence relation and why it partitions $\mathbb{R}^3$ into concentric spheres. (We write $\sim_1$ to distinguish this relation from the one defined in the first part of the problem.)

Solution to Problem 11.9. *We note that all of the sets are nonempty, because they come from partitions. Now $\bigcup_{\alpha \in I \cup J} A_\alpha = \bigcup_{\alpha \in I} A_\alpha \cup \bigcup_{\alpha \in J} A_\alpha = \mathbb{R}^+ \cup \mathbb{R}^- \cup \{0\} = \mathbb{R}$, by the definition of the two partitions. This completes the proof of the second required condition.*

It remains to show that the sets are pairwise disjoint. If we choose sets A_α and A_β from the same partition, then it follows from the definition that they are equal or disjoint. So we may suppose that A_α is from the partition of $\mathbb{R}^+$ and A_β is from the partition of $\mathbb{R}^- \cup \{0\}$. But then

$$A_\alpha \cap A_\beta \subseteq \mathbb{R}^+ \cap (\mathbb{R}^- \cup \{0\}) = \emptyset,$$

and therefore they must be disjoint, completing the proof.

Solution to Problem 11.12. *By Theorem 11.4 the partition induces an equivalence relation $\sim$ on $\mathbb{R}$, where for $x, y \in \mathbb{R}$ we have $x \sim y$ if and only if x and $y \in A$ or x and $y \in B$. This can be reformulated as: $x \sim y$ if and only if both numbers are positive or both are nonpositive.*

Solution to Problem 11.15. *If $A \in \mathcal{A}$, then $A \neq \emptyset$. Hence there is $x \in A \subseteq X$. We claim, that $A = E_x$, where E_x is the equivalence class of x with respect to the equivalence relation $\sim$ induced by $\mathcal{A}$. If $y \in E_x$, then $x \sim y$. So x and y are in the same set in $\mathcal{A}$. Thus, $y \in A$ and hence $E_x \subseteq A$.*
If $y \in A$, then $x \sim y$ by the construction of the equivalence relation. Hence $y \in E_x$ and $A \subseteq E_x$. This shows that $A = E_x$ for some $x \in X$. Thus $A \in \mathcal{B}$ and $\mathcal{A} \subseteq \mathcal{B}$.

Now if $B \in \mathcal{B}$, then $B = E_x$ for some $x \in X$. Since $\mathcal{A}$ partitions X, there is $A \in \mathcal{A}$ with $x \in A$. We claim, that $B = A$.
If $y \in B$, then $y \in E_x$ and thus $x \sim y$. By the construction of $\sim$, there is $C \in \mathcal{A}$ with $x, y \in C$. Hence $A \cap C \neq \emptyset$ and since they are sets of a partition, $A = C$. Thus, $y \in A$ and $B \subseteq A$.
If $y \in A$, then we have $x, y \in A$ and thus $x \sim y$. This implies that $y \in E_x = B$. Hence $A \subseteq B$. This establishes the claim and shows that $\mathcal{B} \subseteq \mathcal{A}$.

The two parts together show that $\mathcal{A} = \mathcal{B}$

Solution to Problem 11.18. *If the partition of X is the trivial partition $\{X\}$, then $\{\mathcal{P}(X)\}$ is the trivial partition of $\mathcal{P}(X)$.*

If the partition $\{A_\alpha : \alpha \in I\}$ of X contains at least two elements, then $\{\mathcal{P}(A_\alpha) : \alpha \in I\}$ does not partition $\mathcal{P}(X)$.
Here is a simple example: $X = \{1, 2\}$ with partition $\{\{1\}, \{2\}\}$. Then $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, X\}$ and the set $\{\mathcal{P}(A_\alpha) : \alpha \in I\}$ is $\{\{\emptyset, \{1\}\}, \{\emptyset, \{2\}\}\}$. The latter set is not a partition of $\mathcal{P}(X)$ because $X \notin \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\}$ and thus condition (ii) is violated.

Solution to Problem 11.21. *We answer both parts below.*

(a) *For $k \in \mathbb{Z}$, we have $k \in A_k$ and thus $A_k \neq \emptyset$. This shows that condition (i) of partition is satisfied.*

Clearly $\bigcup_{k \in \mathbb{Z}} A_k \subseteq \mathbb{Z}$. If $z \in \mathbb{Z}$, then $z \in A_k$ for $k = z \in \mathbb{Z}$. Thus $z \in \bigcup_{k \in \mathbb{Z}} A_k$ and $\mathbb{Z} \subseteq \bigcup_{k \in \mathbb{Z}} A_k$. Hence $\bigcup_{k \in \mathbb{Z}} A_k = \mathbb{Z}$ which is condition (ii) of a partition.

Suppose that for some $r, s \in \mathbb{Z}$ we have $A_r \cap A_s \neq \emptyset$. Then there is $z \in A_r \cap A_s$. Thus $z = 5\ell_1 + r = 5\ell_2 + s$ for some $\ell_1, \ell_2 \in \mathbb{Z}$. Hence $s = 5(\ell_1 - \ell_2) + r$ and $r = 5(\ell_2 - \ell_1) + s$.

We claim that $A_r = A_s$. If $y \in A_s$, then $y = 5\ell + s$ for some $\ell \in \mathbb{Z}$. Using the result from above, we get $y = 5\ell + 5(\ell_1 - \ell_2) + r = 5(\ell + \ell_1 - \ell_2) + r$, where $\ell + \ell_1 - \ell_2 \in \mathbb{Z}$. Hence $y \in A_r$, and we conclude that $A_s \subseteq A_r$. Similarly, $A_r \subseteq A_s$. Thus $A_r = A_s$ and condition (iii) of a partition holds.

We have shown that $\{A_k : k \in \mathbb{Z}\}$ partitions $\mathbb{Z}$.

(b) *Suppose that for $x, y \in \mathbb{Z}$, $x \sim y$. By the construction of Theorem 11.4, $x, y \in A_k$ for some $k \in \mathbb{Z}$ and hence $x = 5\ell_1 + k$ and $y = 5\ell_2 + k$ for some $\ell_1, \ell_2 \in \mathbb{Z}$. Thus $x - y = 5(\ell_1 - \ell_2) = 5n$, where $n \in \mathbb{Z}$.*

Conversely, if $x - y = 5n$ *for some* $n \in \mathbb{Z}$*, then* $x = 5n + y$ *and thus* $x \in A_y$*. Clearly,* $y \in A_y$ *and by the construction of Theorem 11.4,* $x \sim y$*. We have proven that for* $x, y \in \mathbb{Z}$*, we get* $x \sim y$ *if and only if* $x - y$ *is divisible by* 5*.*