Reading, Writing, and Proving (Second Edition)

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Solutions to Chapter 10: Relations

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If you discover errors in these solutions or feel you have a better solution, please write to us at udaepp@bucknell.edu or pgorkin@bucknell.edu. We hope that you have fun with these problems.

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Solution to Problem 10.3. *(a) This is not an equivalence relation because it is not transitive. For example,* $0, 1, 2 \in \mathbb{R}$*,* $0 \sim 1$ *since* $0, 1 \in [0, 1]$*, and* $1 \sim 2$ *since* $1, 2 \in [1, 2]$*. However,* $0 \not\sim 2$*.*

(b) This is an equivalence relation. For reflexivity: Let $x \in \mathbb{R}$*, and let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x*. Then* $x \in (\lfloor x \rfloor, \lfloor x \rfloor + 1)$*, and thus* $x \sim x$*.*

The relation is symmetric because $x, y \in [n, n+1)$ *for some integer* n *if and only if* $y, x \in [n, n+1)$*. Hence* $x \sim y$ *implies that* $y \sim x$*.*

The relation is also transitive. If $x, y, z \in \mathbb{R}$*,* $x \sim y$*, and* $y \sim z$*, then for some* $n \in \mathbb{Z}$*,* $x, y \in [n, n+1)$*; that is,* $n \leq x < n+1$ *and* $n \leq y < n+1$*. Hence* $n = \lfloor y \rfloor$*. Also, for some* $m \in \mathbb{Z}$*,* $y, z \in [m, m+1)$ *which means that* $m \leq y, z < m+1$*. We conclude that* $m = \lfloor y \rfloor$*. Thus* $m = n$ *and* $x, z \in [n, n+1)$*, that is,* $x \sim z$*.*

(c) This is not an equivalence relation. We will show that it is not transitive: $0, 1.5, 2.5 \in \mathbb{R}$*.* $0 \sim 1.5$*,* $1.5 \sim 2.5$*, but* $0 \not\sim 2.5$*.*

Solution to Problem 10.6. *Say* $x \sim y$ *if* x, y *are both positive, both negative or both zero. Then this is an equivalence relation and the equivalence classes are (as you should check)* $\mathbb{Z}^+, \{0\}, \mathbb{Z}^-$*.*

Solution to Problem 10.9. *This is an equivalence relation. First,* $(x, y) \sim (x, y)$ *for all* $(x, y) \in \mathbb{R}^2$ *because* $x - x = y - y = 0$ *is an even integer. So this relation is reflexive.*

Now suppose that (x, y) *and* (w, z) *are elements of* $\mathbb{R}^2$ *with* $(x, y) \sim (w, z)$*. Then* $x - w \in 2\mathbb{Z}$ *and* $y - z \in 2\mathbb{Z}$*. Therefore* $w - x \in 2\mathbb{Z}$ *and* $z - y \in 2\mathbb{Z}$*. Consequently,* $(w, z) \sim (x, y)$ *and this relation is symmetric.*

Finally, suppose $(a, b), (c, d), (f, g) \in \mathbb{R}^2$ *with* $(a, b) \sim (c, d)$ *and* $(c, d) \sim (f, g)$*. Then* $a - c, b - d, c - f$ *and* $d - g$ *are all even integers. Therefore,*

$$a - f = (a - c) + (c - f) \text{ and } b - g = (b - d) - (d - g)$$

are both even integers. Thus $(a, b) \sim (f, g)$ *and the relation is transitive.*

Since all the relation is reflexive, symmetric, and transitive, the relation is an equivalence relation.

Solution to Problem 10.12. *This is really a great problem, so it would be a pity to read the solution without giving it the old college try first. If you are having trouble thinking of an example, Problem 10.3 might suggest something.*

One relation that works is the following:

$$(r, s) \sim (u, v) \text{ if and only if } |r - u| + |s - v| \leq 1.$$

Then it should be clear that $(r, s) \sim (r, s)$ *for all* $(r, s) \in \mathbb{Z} \times \mathbb{Z}$ *and* $(r, s) \sim (u, v)$ *implies that* $(u, v) \sim (r, s)$ *for all* $(r, s), (u, v) \in \mathbb{Z} \times \mathbb{Z}$. *So it should be clear that the relation is reflexive and symmetric.*

As for transitivity, note that $(0, 0) \sim (1, 0)$ *and* $(1, 0) \sim (1, 1)$ *but* $(0, 0)$ *is not related to* $(1, 1)$. *Therefore, the relation is not transitive.*

To test your understanding of this solutions, you might try thinking of a different relation on $\mathbb{Z} \times \mathbb{Z}$ *with the same properties.*

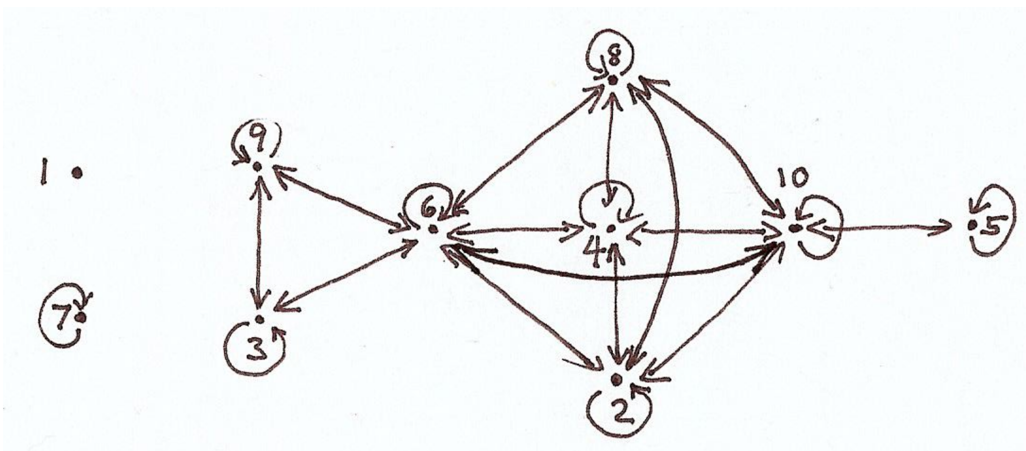

Figure 10.1:

Solution to Problem 10.15. *(a) Figure 10.1 shows the diagram for this relation. This relation is not reflexive, since* 1 *has no arrow from and to itself. It is symmetric because every arrow between different vertices goes in both directions. It is not transitive, e.g there are arrows from* 3 *to* 6 *and from* 6 *to* 2 *but there is no direct arrowfrom* 3 *to* 2. *Hence this relation is not an equivalence relation.*

(b) Figure 10.2 shows the diagram of this relation. Every vertex has an arrow from and to itself, so the relation is reflexive. All arrows between two different vertices go both ways, hence the relation is symmetric. Finally, whenever an arrow goes from one vertex to a second one and another arrow goes from the second to a third vertex, then there is an arrow going directly from the first to the third arrow. This shows that the relation is an equivalence relation.

The diagram has three components, hence there are three equivalence classes. They are $E_4 = \{4, 6, 8, 12\}$, $E_5 = \{5, 35\}$, *and* $E_{11} = \{11, 143\}$.

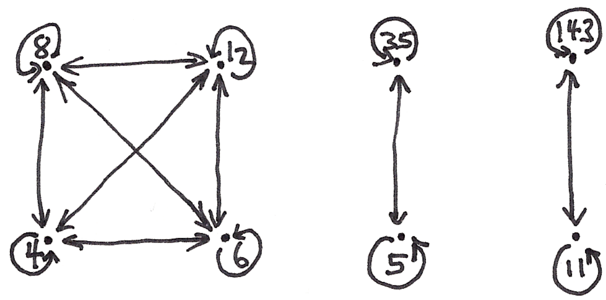

Figure 10.2:

Solution to Problem 10.18. *If $z \in \bigcup_{x \in X} E_x$, then $z \in E_x$ for some $x \in X$. Since $E_x \subseteq X$ we get $z \in X$ and thus $\bigcup_{x \in X} E_x \subseteq X$. For the converse, if $z \in X$, then, since $z \sim z$, we have $z \in E_z$ and $z \in X$. Thus $z \in \bigcup_{x \in X} E_x$. Hence $X \subseteq \bigcup_{x \in X} E_x$. The two inclusions show that $\bigcup_{x \in X} E_x = X$.*