MATH 280 **TEST 1** **9/20/2010**

Name: ______________________________ Points: ________/100

Work all five problems very carefully showing all steps in your solutions. Each problem is worth 20 points.

Problem 1. *Consider the statement form* $(R \wedge P) \rightarrow ((\neg Q \vee P) \wedge (R \vee Q))$.

(a) Write out the truth table of this statement form. Show the intermediate steps. What can you conclude?

(b) Write the contrapositive statement form. Do you expect the truth table to be the same or different. (Do NOT calculate it.) Give a brief reason for your answer.

Problem 2. *Consider the statement* $\forall x, ((\exists y, x = y) \rightarrow \exists z, (x = z^2 \vee y \neq z^2))$

(a) Negate this statement.

(b) Take $\mathbb{R}$ *for the universe of* x *and* $\mathbb{Z}$ *for the universe of* y *and* z*. Using these universes write the original statement as an English sentence. First give a verbatim "translation," then simplify it and make it as eloquent as you can.*

Problem 3. *Let* $A, B,$ *and* C *be subsets of a universe* X.

(a) Draw a Venn diagram for $(A \cup (B \setminus C)) \cap (X \setminus (C \cup (A \setminus B)))$ *and another one for* $(B \setminus A) \cap (C \cup A)^c$.

(b) Give one condition on two of the three sets that would make the two configurations in (a) equal. (You need not prove anything here.)

Problem 4. *Let* $A, B,$ *and* C *be subsets of a universe* X.
Prove that $A \cap (B^c \cap C^c) = \emptyset$ *if and only if* $A \subseteq B \cup C$.

Problem 5. *We define the following collections of subsets of* $\mathbb{Q}$.

$$A_n = \left\{ \frac{p}{2^n} : p \in \mathbb{Z} \right\} \quad \text{and} \quad B_m = \left\{ \frac{q}{2^m(2t+1)} : q \in \mathbb{Z}, t \in \mathbb{Z}^+, \text{ and } (2t+1) \not| \, q \right\}$$

Consider the following set equation.

$$\bigcap_{n \in \mathbb{N}} (\mathbb{Q} \setminus A_n) = \bigcup_{m \in \mathbb{N}} B_m \tag{1}$$

(a) Which ones of the real numbers $4, 0, \frac{3}{8}, \frac{15}{12}, \frac{5}{60},$ *and* $\frac{13}{15}$ *are in* $\bigcup_{m \in \mathbb{N}} B_m$?

(b) Prove equation (1).

MATH 280 **TEST 2** **10/25/2010**

Name: ______________________________ Points: ________/100

Work all four problems very carefully showing all steps in your solutions. Each problem is worth 25 points.

Problem 1. *Consider the object a "function f from set A to set B."*

(a) *Define this object.*

(b) *Give an example of a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathcal{P}(\mathbb{N})$. You need NOT prove that your example is a function.*

(c) *Give an example of an object from $\mathbb{R}$ to $\{1, 2, 3\}$ that violates exactly one of the conditions for functions that you stated in your definition and satisfies all the other ones. Again, you need NOT prove anything here but you should indicate which one property is violated.*

Problem 2. *For $x \in \mathbb{R} \setminus \{0\}$, let $A_x = \{y \in \mathbb{R} \setminus \{0\} : xy > 0\}$.*

(a) *Prove that $\mathcal{A} = \{A_x : x \in \mathbb{R} \setminus \{0\}\}$ is a partition of $\mathbb{R} \setminus \{0\}$.*

(b) *Denote by $\sim$ the equivalence relation that is associated with this partition. Give a simple description of $x \sim y$ for $x, y \in \mathbb{R} \setminus \{0\}$. You need not prove that your description is correct.*

(c) *We cannot define a function $f : \mathcal{A} \to \mathbb{R}$ by $f(A_x) = x$. Why not?*

Problem 3.

Definition. *Let A and B be two nonempty sets and $f : A \to B$ a function. Then a function $g : B \to A$ is called a left inverse for f if $g \circ f = i_A$.*

(a) *Prove that f has a left inverse if and only if f is injective (one-to-one).*

(b) *Give an example of a function f that is surjective (onto) and does not have a left inverse. Give a quick reason why your example works.*

(c) *Give an example of a function that has a left inverse but no inverse. Give a quick reason why your example works.*

Problem 4. *Let A and B be two nonempty subsets of $\mathbb{R}$ such that $A \cup B = \mathbb{R}$ and for all $a \in A$ and $b \in B$ we have $a < b$.*

(a) *Prove that $\sup A$ and $\inf B$ exist. (This just requires the quotation of the relevant fact and the quick verification that the required conditions are met.)*

(b) *Prove that $\sup A = \inf B$.*

MATH 280 **TEST 3** **11/22/2010**

Name: ____________________ Points: ______/100

Work all four problems very carefully showing all steps in your solutions. Each problem is worth 25 points.

Problem 1. *Prove that for real numbers a and r, with $r \neq 1$ we have*

$$\sum_{j=0}^{n} ar^j = \frac{a(1-r^{n+1})}{1-r} \quad \text{for all } n \in \mathbb{N}.$$

Problem 2. *(a) Define what it means for a sequence to converge.*

(b) Give an example of an increasing sequence that converges and an example of an increasing sequence that diverges. You need not prove that your examples work.

(c) Using your definition from part (a) only, prove that the sequence (x_n) converges, where $x_n = \frac{n+n^2}{1-n+n^2}$.

Problem 3. *Denote by $\mathcal{F}$ the set of all functions $f : \mathbb{N} \to \mathbb{N}$.*

(a) Give two examples of elements g_1 and g_2 of $\mathcal{F}$

(b) Using the functions g_1 and g_2 that you defined in part (a), give an example of an element ℓ of $\mathcal{F}$ such that $\ell(0) \neq g_1(0)$ and $\ell(1) \neq g_2(1)$.

(c) Let $\mathcal{A}$ be a countable subset of $\mathcal{F}$. That is, $\mathcal{A} = \{f_0, f_1, f_2, f_3, f_4, \ldots : f_j \in \mathcal{F}\}$. Define a function $h \in \mathcal{F}$ such that $h \notin \mathcal{A}$.

(d) Prove that $\mathcal{F}$ is uncountable.

Problem 4. *For this problem use definitions and theorems from the book or from the lectures through Chapter 23 only.* ***In particular, you may use the fact that the Cartesian product of two finite sets is finite but you may NOT use the fact that if there exists $f : X \to Y$ injective, then $|X| \leq |Y|$.*** *Also, you cannot use facts of problems you may have worked—nor any rules based on intuitive ideas only.*

Let A be a nonempty finite set. Prove that $|A| \leq |A \times A|$.

MATH 280-01 **FINAL EXAMINATION** **12/14/2010**

Name: ______________________________ Points: ________/150

Each problem is worth 25 points. **You may use any theorem from the book or the class that you remember but you cannot use results from problems you solved.**

Problem 1. *For an integer x consider the statement: "If 8 does not divide $x^2 - 1$, then x is even."*

(a) *State an appropriate universe for x.*

(b) *Write the statement in symbols. (No words at all, translate "even" also into symbols. Make sure you write the quantifiers!)*

(c) *Negate the statement (leaving the answer in symbols).*

(d) *Prove the statement that is true.*

Problem 2. (a) *Define* ***power set***.

Let $\mathcal{A} = \{A_\alpha : \alpha \in I\}$ and $\mathcal{B} = \{B_\beta : \beta \in J\}$ be two indexed collections of sets that have the properties that (1) to every $\alpha \in I$, there is some $\beta \in J$ such that $A_\alpha \subseteq B_\beta$; and (2) to every $\beta \in J$ there is some $\alpha \in I$ such that $B_\beta \subseteq A_\alpha$.

(b) *Prove that*

$$\mathcal{P}\left(\bigcap_{\alpha \in I} A_\alpha\right) \subseteq \bigcup_{\beta \in J} \mathcal{P}(B_\beta).$$

(c) *Give an example of two collections of sets, $\mathcal{A}$ and $\mathcal{B}$, that satisfy both of the properties above, but*

$$\mathcal{P}\left(\bigcap_{\alpha \in I} A_\alpha\right) \neq \bigcup_{\beta \in J} \mathcal{P}(B_\beta).$$

Problem 3. (a) *Define:* ***a sequence*** $(\mathbf{x_n})$ ***converges***.

(b) *Use this definition only to prove that if a sequence (x_n) of positive terms converges to a real number $L > 0$, then the sequence $(\sqrt{x_n})$ converges to $\sqrt{L}$.*

Problem 4. *Let $f : \mathbb{R} \setminus \{2\} \to \mathbb{R}$ be defined by $f(x) = 3x/(x-2)$. Let $h : \mathbb{R} \to (0,1)$ be a bijection.*

(a) *Is $h \circ f : \mathbb{R} \setminus \{2\} \to (0,1)$ one-to-one? Prove it or show why it isn't.*

(b) *Is $h \circ f : \mathbb{R} \setminus \{2\} \to (0,1)$ onto? Prove it or show why it isn't.*

Problem 5. *Prove that $3^n - n^2 > 2^n + n$ for all integers n with $n \geq 3$.*

Problem 6. *We denote by $\mathcal{F}$ the collection of all finite subsets of $\mathbb{Z}^+$. Further, p_j denotes the j-th prime number. (So $p_1 = 2, p_2 = 3, \ldots$.)*

(a) *We define*

$$g : \mathcal{F} \to \mathbb{Z}^+, \quad \text{by} \quad g(X) = \prod_{j \in X} p_j \ \text{ for } X \in \mathcal{F} \text{ with } X \neq \emptyset \text{ and } g(\emptyset) = 1.$$

Prove that this is a well-defined function that is injective.

(b) *Find an injective function $h : \mathbb{Z}^+ \to \mathcal{F}$ (if necesary, prove that your function is well-defined and prove that your function is injective).*

(c) *Using parts (a) and (b) (even if you were not able to show them completely), prove that $|\mathcal{F}| = |\mathbb{Z}^+|$. State the theorem that you use to establish this equality.*