Fall 2009 Eastern Sectional Meeting of the AMS
Special session on Algebraic Combinattorics

Vertex decomposable graphs and obstructions to shellability

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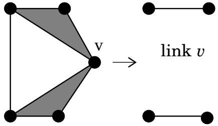

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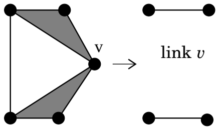

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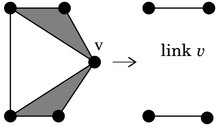

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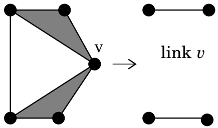

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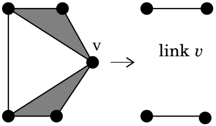

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Vertex decomposable $\implies$ Shellable $\implies$ seq. Cohen-Macaulay

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Basic notions

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Approach: Examine graph theoretic properties of G and their consequences for the independence complex.

Basic notions – dictionary

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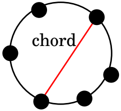
Chordal graphs are vertex decomposable

A graph is *chordal* if it contains no induced cycles of length > 3.

Equivalently, every cycle of length ≥ 4 has a "chord".



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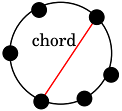

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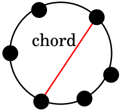

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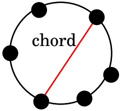

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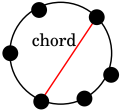

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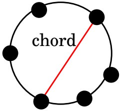

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Theorem: (me) If for every independent A in a graph G the subgraph $G \setminus N[A]$ has a "simplicial vertex", then the independence complex of G is vertex decomposable.

Chordal graphs are vertex decomposable – sketch

Shedding vertex v: independent sets of $G \setminus N[v]$ are not maximal independent sets of $G \setminus v$.

Main fact: If G is chordal, then G has vertex w with $N[w]$ a complete subgraph.
Such a w is called a *simplicial vertex*.

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To show that every link has simplicial vertex $\implies$ vertex dec., notice that repeated deletion of neighbors of w leaves $w \dot\cup G \setminus N[w]$.

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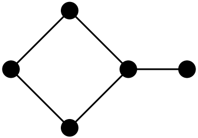

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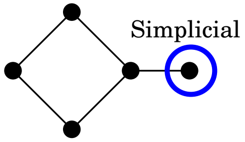

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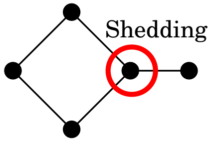

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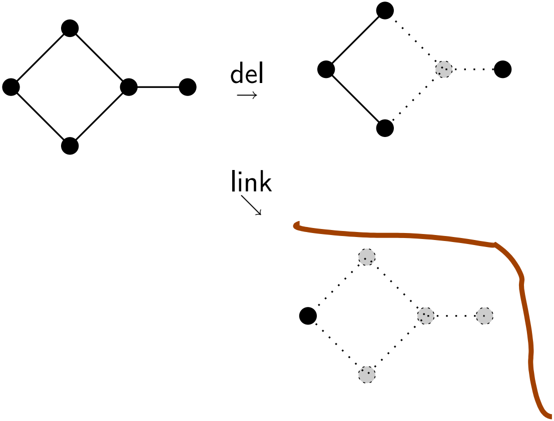

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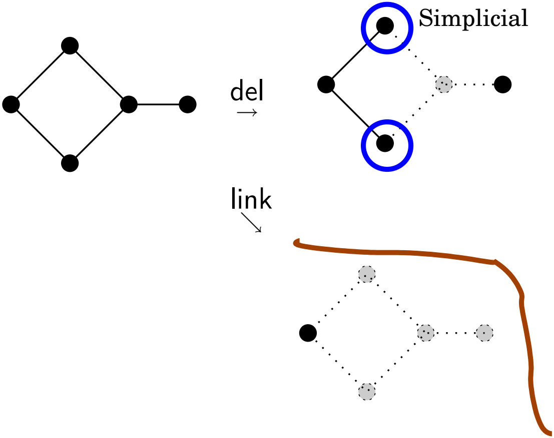

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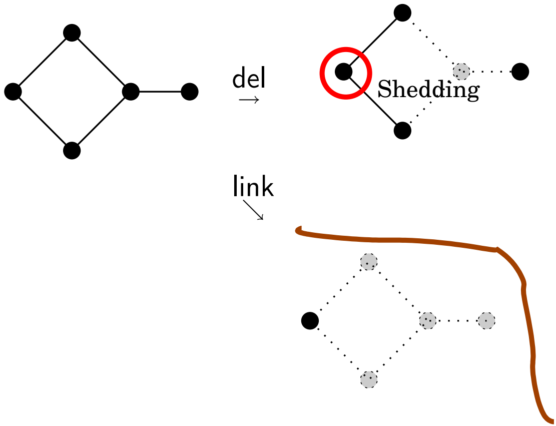

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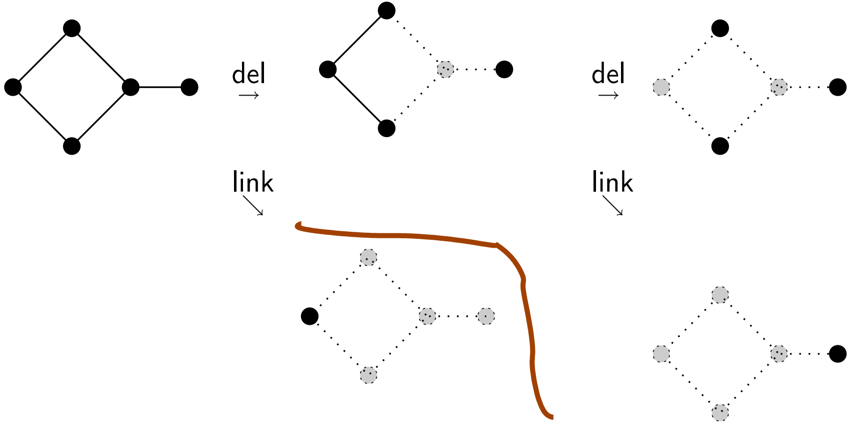

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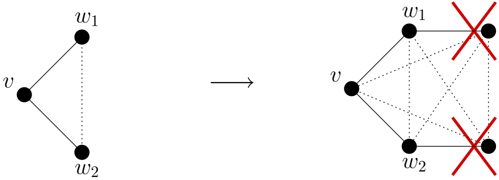

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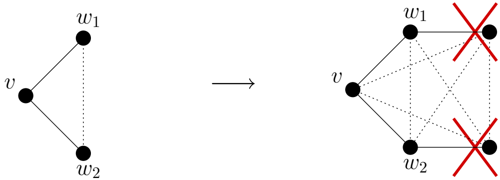

The middle vertex v of a 3-simplicial path is a shedding vertex:

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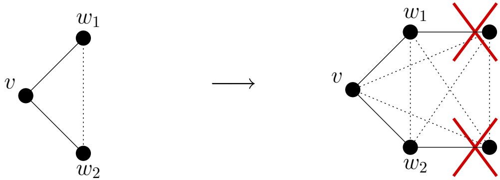
Theorem: (me) If G contains no induced cycles of length other than 3 or 5, then G is vertex decomposable.

Sketch: A non-trivial theorem of Chvátal, Rusu, and Sritharan says that a graph with no cycles ≥ 6 which is not the disjoint union of complete graphs has a "3-simplicial path". This is a path of length 3 that does not sit inside any chordless path of length 5.



The middle vertex v of a 3-simplicial path is a shedding vertex: An independent set in $G \setminus N[v]$ can be augmented by either w_1 or w_2, since it can't neighbor both of them.

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Obstructions to shellability

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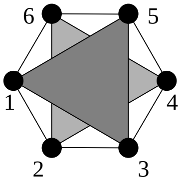

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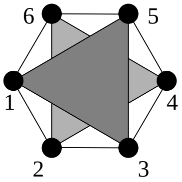
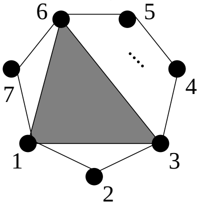


C_7:



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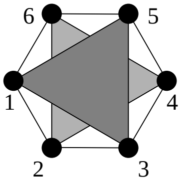

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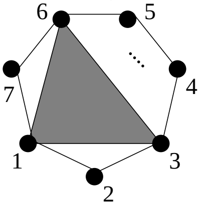

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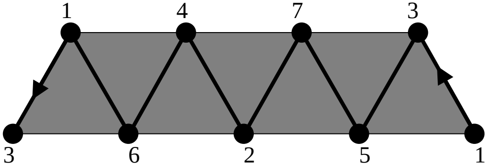

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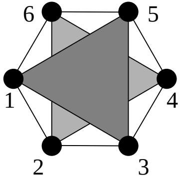

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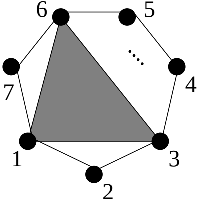

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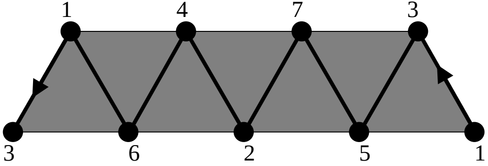

Corollary: (me) The obstructions to shellability (minimal non-shellable complexes) in flag complexes are exactly the independence complexes of C_n, $n \neq 3, 5$.

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Can we relate the clutter-theoretic properties of $\mathcal{C}$ to shellability of its independence complex?

Chordal clutters

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Theorem: (me) The independence complex of a chordal clutter is shellable.

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Application: there are 21 obstructions to shellability on 6 vertices that have every link shellable. (by GAP computation)

Reference:

1. Russ Woodroofe, *Vertex decomposable graphs and obstructions to shellability*, Proc. Amer. Math. Soc. 137 (2009), no. 10, 3235–3246, arXiv:0810.0311.
2. Russ Woodroofe, *Chordal and sequentially Cohen-Macaulay clutters*, on my webpage http://www.math.wustl.edu/∼russw/

Thank you!

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