On the stability of the Kronecker product

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http://arxiv.org/pdf/0907.4652

The Clebsch-Gordan Problem

Decompose into irreducibles the tensor product of two irreducible representations of a (finite reductive) group G:

$$V_\mu \otimes V_\nu = \bigoplus_\lambda m^\lambda_{\mu,\nu} V_\lambda$$

Examples:

- **The general linear groups,** $GL_n(\mathbb{C})$: The multiplicities, $m^\lambda_{\mu,\nu}$ are the Littlewood-Richardson coefficients $c^\lambda_{\mu,\nu}$.
- **The symmetric groups,** $\mathfrak{S}_n$: The multiplicities $m^\lambda_{\mu,\nu}$ are the Kronecker coefficients $g^\lambda_{\mu,\nu}$.

Kronecker product of symmetric functions

The Frobenious map identifies irreducible representation V_λ of the symmetric group with the Schur function s_λ. Accordingly, the Kronecker product of symmetric functions can be defined by

$$s_\mu * s_\nu = \sum_\lambda g^\lambda_{\mu,\nu} s_\lambda$$

Stability of the Kronecker product

Murnaghan (1938, 1955) observed that the Kronecker product of Schur functions $s_\mu * s_\nu$ stabilizes when incrementing the first parts of λ and μ.

Example:

$s_{2,2} * s_{2,2} = s_4 + s_{1,1,1,1} + s_{2,2}$
$s_{3,2} * s_{3,2} = s_5 + s_{2,1,1,1} + s_{3,2} + s_{4,1} + s_{3,1,1} + s_{2,2,1}$
$s_{4,2} * s_{4,2} = s_6 + s_{3,1,1,1} + 2s_{4,2} + s_{5,1} + s_{4,1,1} + 2s_{3,2,1} + s_{2,2,2}$
$s_{5,2} * s_{5,2} = s_7 + s_{4,1,1,1} + 2s_{5,2} + s_{6,1} + s_{5,1,1} + 2s_{4,2,1} + s_{3,2,2} + s_{4,3} + s_{3,3,1}$
$s_{6,2} * s_{6,2} = s_8 + s_{5,1,1,1} + 2s_{6,2} + s_{7,1} + s_{6,1,1} + 2s_{5,2,1} + s_{4,2,2} + s_{5,3} + s_{4,3,1} + s_{4,4}$
$s_{7,2} * s_{7,2} = s_9 + s_{6,1,1,1} + 2s_{7,2} + s_{8,1} + s_{7,1,1} + 2s_{6,2,1} + s_{5,2,2} + s_{6,3} + s_{5,3,1} + s_{5,4}$
$s_{\bullet,2} * s_{\bullet,2} = s_\bullet + s_{\bullet,1,1,1} + 2s_{\bullet,2} + s_{\bullet,1} + s_{\bullet,1,1} + 2s_{\bullet,2,1} + s_{\bullet,2,2} + s_{\bullet,3} + s_{\bullet,3,1} + s_{\bullet,4}$

Murnaghan's Theorem

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, then $\alpha[n] = (n - |\alpha|, \alpha_1, \ldots, \alpha_k)$.

Theorem: There exists a family of non-negative integers $(\bar{g}^{\gamma}_{\alpha\beta})$ indexed by triples of partitions (α, β, γ) such that, for α and β fixed, only finitely many terms $\bar{g}^{\gamma}_{\alpha\beta}$ are nonzero, and for all $n \geq 0$,

$$s_{\alpha[n]} * s_{\beta[n]} = \sum_{\gamma} \bar{g}^{\gamma}_{\alpha\beta} s_{\gamma[n]}$$

Note: $s_\mu = det(h_{\mu_i+i-j})_{1\leq,i,j\leq n}$.

Example:

$s_{2,2} * s_{2,2} = s_4 + s_{1,1,1,1} + 2s_{2,2} + s_{3,1} + s_{2,1,1} + 2s_{1,2,1} + s_{0,2,2} + s_{1,3} + s_{0,3,1} + s_{0,4}$

Reduced Kronecker coefficients

Definition: Given any three partitions, α, β, and γ, the sequence

$$(g^{\gamma[n]}_{\alpha[n],\beta[n]})$$

is eventually constant. The **reduced Kronecker coefficient** $\bar{g}^{\gamma}_{\alpha,\beta}$ is defined as the stable value of this sequence.

Example:

$$(g^{(k+1,2,2,1,1)}_{(k,3,2,1,1)(k,3,2,2)}) = (17, 119, 256, 305, 308, 308, \ldots)$$

and

$$\bar{g}^{(2,2,1,1)}_{(3,2,1,1),(3,2,2)} = 308$$

Kronecker coefficients from the reduced Kronecker coefficients

Let $\bar{\lambda} = (\lambda_2, \lambda_3, \ldots) = \lambda^{\dagger 1}$

Theorem:

$$g_{\mu,\nu}^{\lambda} = \sum_{i=1}^{\ell(\mu)\ell(\nu)} (-1)^{i+1} \bar{g}_{\bar{\mu}\bar{\nu}}^{\lambda^{\dagger i}}$$

where $\lambda^{\dagger i}$ is the partition obtained by removing the i-th part and adding 1 to the first $i-1$ parts.

Example:

$$g_{(\mu_1,\mu_2)(\nu_1,\nu_2)}^{(\lambda_1,\lambda_2,\lambda_3)} = \bar{g}_{(\mu_2)(\nu_2)}^{(\lambda_2,\lambda_3)} - \bar{g}_{(\mu_2)(\nu_2)}^{(\lambda_1+1,\lambda_3)} + \bar{g}_{(\mu_2)(\nu_2)}^{(\lambda_1+1,\lambda_2+1)} - \bar{g}_{(\mu_2)(\nu_2)}^{(\lambda_1+1,\lambda_2+1,\lambda_3+1)}$$

Stabilization of $s_{\alpha[n]} * s_{\beta[n]}$

Question: It is natural to ask about the index n at which the expansion of $s_{\alpha[n]} * s_{\beta[n]}$ stabilizes.

Definition: Let V be the linear operator on symmetric functions defined on the Schur basis by $V(s_\lambda) = s_{\lambda+(1)}$ for all partitions λ. Let α and β be partitions. Then stab(α, β) is defined as the smallest integer n such that $s_{\alpha[n+k]} * s_{\beta[n+k]} = V^k(s_{\alpha[n]} * s_\beta[n])$ for all $k > 0$.

Example: Let $\alpha = \beta = (2)$ then stab$(\alpha, \beta) = 8$.

Value of **stab**(α, β)

Theorem: Let α and β be partitions. Then

$$\mathrm{stab}(\alpha, \beta) = |\alpha| + |\beta| + \alpha_1 + \beta_1$$

To prove this theorem we first show that

$$\mathrm{stab}(\alpha, \beta) = \max\left\{ |\gamma| + \gamma_1 \,|\, \gamma \text{ partition s.t. } \bar{g}_{\alpha\beta}^{\gamma} > 0 \right\}$$

The width of γ for Kronecker coefficients

$$\alpha \cap \beta = (\min(\alpha_1, \beta_1), \min(\alpha_2, \beta_2), \ldots)$$

Theorem: [Klemm, Dvir, Clausen-Meier] Let α and β be partitions with the same weight. Then

$$\max\left\{\gamma_1 \,|\, \gamma \text{ partition s.t. } g_{\alpha\beta}^{\gamma} > 0\right\} = |\alpha \cap \beta|.$$

The width of γ for reduced Kronecker coefficients

Theorem: Let α and β be partitions. Then

$$\max\left\{\gamma_1 \mid \gamma \text{ partition s.t. } \bar{g}_{\alpha\beta}^{\gamma} > 0\right\} = |\alpha \cap \beta| + \max(\alpha_1, \beta_1).$$

We use the following more general theorem for part of the proof:

Let $E_i\alpha$ denote the partition obtained by removing the i-th part from α.

Theorem: Let α and β be partitions. Then

$$\gamma_{i+j-1} \leq |E_i\alpha \cap E_j\beta| + \alpha_i + \beta_j.$$

Max and Min for $|\gamma|$

Theorem: Let α and β be partitions. We have

$$\begin{aligned} \max\left\{|\gamma| \mid \gamma \text{ partition s.t. } \bar{g}_{\alpha\beta}^{\gamma} > 0\right\} &= |\alpha| + |\beta| \\ \min\left\{|\gamma| \mid \gamma \text{ partition s.t. } \bar{g}_{\alpha\beta}^{\gamma} > 0\right\} &= \max(|\alpha|, |\beta|) - |\alpha \cap \beta| \end{aligned}$$

Bounds for the rows of γ

Corollary: let α and β be partitions and i and j positive integers such that $k = i + j - 1$. Then

$$\max\left\{\gamma_k \,|\, \gamma \text{ partition s.t.} \bar{g}_{\alpha\beta}^{\gamma} > 0\right\} \leq \min\left(|E_i\alpha \cap E_j\beta| + \alpha_i + \beta_j, \left\lfloor\frac{|\alpha| + |\beta|}{k}\right\rfloor\right)$$

Example

Let $\alpha = (2)$ and $\beta = (4, 3, 2)$, then the first row of the table are the nonzero values of γ_k and the second row are the upper bounds given by the Corollary

k	1	2	3	4	5
max values for γ_k	6	4	3	2	1
bound for γ_k	6	5	3	2	2

In the case that $\alpha = (3, 1)$ and $\beta = (2, 2)$ we get

k	1	2	3	4	5	6
max values for γ_k	6	3	2	1	1	1
bound for γ_k	6	4	2	2	1	1

About the proofs

$$c^{\delta}_{\alpha,\beta,\gamma} = \sum_{\varphi} c^{\varphi}_{\alpha,\beta} c^{\delta}_{\varphi,\gamma} \qquad (1)$$

Lemma: Let α, β, γ be partitions. Then $\overline{g}^{\gamma}_{\alpha,\beta}$ is positive if and only if there exist partitions δ, ϵ, ζ, ρ, σ, τ such that all four coefficients $g^{\zeta}_{\delta,\epsilon}$, $c^{\alpha}_{\delta,\sigma,\tau}$, $c^{\beta}_{\epsilon,\rho,\tau}$ and $c^{\gamma}_{\zeta,\rho,\sigma}$ are positive. Moreover,

$$\overline{g}^{\gamma}_{\alpha,\beta} = \sum g^{\zeta}_{\delta,\epsilon} c^{\alpha}_{\delta,\sigma,\tau} c^{\beta}_{\epsilon,\rho,\tau} c^{\gamma}_{\zeta,\rho,\sigma} \qquad (2)$$

Stability of Kronecker coefficients

From Murnaghan's Theorem we know that each particular sequence of Kronecker coefficients $g_{\alpha[n],\beta[n]}^{\gamma[n]}$ stabilizes with value $\overline{g}_{\alpha,\beta}^{\gamma}$, possibly before reaching stab(α, β)

Definition: Let α, β, γ be partitions. Then stab(α, β, γ) is defined as the the smallest integer N such that the sequences $\alpha[N]$, $\beta[N]$ and $\gamma[N]$ are partitions and $g_{\alpha[n],\beta[n]}^{\gamma[n]} = \overline{g}_{\alpha,\beta}^{\gamma}$ for all $n \geq N$.

Problem: What is stab(α, β, γ) in terms of α, β and γ?

Brion's and Vallejo's bounds

$$M_B(\alpha,\beta;\gamma) = |\alpha| + |\beta| + \gamma_1,$$

$$M_V(\alpha,\beta;\gamma) = |\gamma| + \begin{cases} \max\{|\alpha| + \alpha_1 - 1, |\beta| + \beta_1 - 1, |\gamma|\} & \text{if } \alpha \neq \beta \\ \max\{|\alpha| + \alpha_1, |\gamma|\} & \text{if } \alpha = \beta \end{cases}$$

Then, Brion's bounds is

$$N_B = \min\{M_B(\alpha,\beta;\gamma), M_B(\alpha,\gamma;\beta), M_B(\gamma,\beta;\alpha)\}$$

and Vallejo's

$$N_V = \min\{M_V(\alpha,\beta;\gamma), M_V(\alpha,\gamma;\beta), M_V(\gamma,\beta;\alpha)\}$$

Technique for finding bounds for $stab(\alpha, \beta, \gamma)$

Lemma: Let f be a function on triples of partitions such that for all i,

$$f(\alpha, \beta, \bar{\gamma}) \geq f(\alpha, \beta, \gamma^{\dagger i}).$$

Set

$$\mathcal{M}_f(\alpha, \beta, \gamma) = |\gamma| + f(\alpha, \beta, \bar{\gamma})$$

and assume also that whenever $\overline{g}^{\gamma}_{\alpha,\beta} > 0$,

$$\mathcal{M}_f(\alpha, \beta, \gamma) \geq \max\left(|\alpha| + \alpha_1, |\beta| + \beta_1, |\gamma| + \gamma_1\right).$$

Then whenever $\overline{g}^{\gamma}_{\alpha,\beta} > 0$,

$$\mathrm{stab}(\alpha, \beta, \gamma) \leq \mathcal{M}_f(\alpha, \beta, \gamma).$$

Examples of functions f in Lemma

(1) $f(\alpha, \beta, \tau) = |\alpha| + |\beta| - |\tau|$.

(2) $f(\alpha, \beta, \tau) = |\bar{\alpha} \cap \bar{\beta}| + \alpha_1 + \beta_1$.

(3) $f(\alpha, \beta, \tau) = \frac{1}{2}(|\alpha| + |\beta| + \alpha_1 + \beta_1 - |\tau|)$.

Remark: Using the first function and our lemma we obtain Brion's bound.

Our First Bound for **stab**(α, β, γ)

Using $f(\alpha, \beta, \tau) = |\bar{\alpha} \cap \bar{\beta}| + \alpha_1 + \beta_1$ and the theorem on the width of the reduced Kronecker coefficients we obtain:

Theorem: Let $M_1(\alpha, \beta; \gamma) = |\gamma| + |\bar{\alpha} \cap \bar{\beta}| + \alpha_1 + \beta_1$ and

$$N_1(\alpha, \beta, \gamma) = \min \{M_1(\alpha, \beta; \gamma), M_1(\alpha, \gamma; \beta), M_1(\beta, \gamma; \alpha)\}$$

Then

$$\text{stab}(\alpha, \beta, \gamma) \leq N_1(\alpha, \beta, \gamma).$$

Our Second Bound for **stab**(α, β, γ)

Using $f(\alpha, \beta, \tau) = \frac{1}{2}(|\alpha| + |\beta| + \alpha_1 + \beta_1 - |\tau|)$ and the theorem for stab(α, β) we obtain:

Theorem: Let

$$N_2(\alpha, \beta, \gamma) = \left[\frac{|\alpha| + |\beta| + |\gamma| + \alpha_1 + \beta_1 + \gamma_1}{2} \right]$$

where $[x]$ denotes the integer part of x. Then

$$\textbf{stab}(\alpha, \beta, \gamma) \leq N_2(\alpha, \beta, \gamma).$$

N_1 is better than N_B and N_V

Proposition: Let α, β, γ be partitions, then $N_1(\alpha, \beta, \gamma) \leq N_B(\alpha, \gamma, \beta)$ and $N_1(\alpha, \beta, \gamma) \leq N_V(\alpha, \beta, \gamma)$.

Comparing N_2 to N_B and N_V

- Let $\alpha = (2,1)$ and $\beta = (3,1)$,
  if $\gamma = (3,1)$, then $N_B = 10 > N_2 = 9$ and
  if $\gamma = (3,2,2)$ then $N_B = 10 < N_2 = 11$.

- Let $\alpha = (2,1)$, $\beta = (3,1)$ and $\gamma = (3,2,2)$, then
  $N_2 = 11 < N_V = 12$.
  If $|\alpha| = |\beta|$ with $\alpha_1 = \beta_1$ and $\gamma = (\gamma_1)$, then $N_V \leq N_2$.

Another Example

If $\alpha = (3,2)$, $\beta = (2,2,1)$, $\gamma = (2,2)$, then $\mathrm{stab}(\alpha,\beta,\gamma) = 10$, but

$$N_B(\alpha,\beta,\gamma) = N_V(\alpha,\beta,\gamma) = N_1(\alpha,\beta,\gamma) = 11.$$

But, $N_2(\alpha,\beta,\gamma) = 10$.

Thanks!