Combinatorics of affine K-theory

- Classical setting

  Schur and Grothendieck

- Affine setting

  k-Schur and their dual

  Affine Grothedieck and K-k-Schur

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Schur functions: s_λ

Symmetric functions

Basis for $\Lambda = \mathbb{Z}[h_1, h_2, \ldots,]$

Indexed by partitions

Homogeneous degree

$$s_{(2,1)} = 2x_1x_2x_3 + x_1x_1x_2 + x_2x_2x_3 + x_1x_1x_3 + x_1x_2x_2 + x_1x_3x_3 + x_2x_3x_3$$

Schur functions: s_λ

Kostka Numbers

tensor product multiplicities of general linear group

$$h_\lambda = \sum_\mu K_{\lambda\mu}\, s_\mu$$

Pieri rule

determines product on cohomology of the Grassmannian

$$s_r\, s_\lambda = \sum s_\nu$$

Littlewood-Richardson coefficients

structure constants for cohomology of Grassmannian

tensor product multiplicities of general linear group

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu}\, s_\nu$$

Combinatorics of Schurs

Definition

$$\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} + \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} + \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} + \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} + \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} + \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$$

$$S_{(2,1)} = x_1x_2x_3 + x_1x_2x_3 + x_1x_1x_2 + x_2x_2x_3 + x_1x_1x_3 + x_1x_2x_2 + x_1x_3x_3 + x_2x_3x_3$$

Combinatorics of Schurs

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Kostka Numbers

$$h_\lambda = \sum_\mu (\#\ \text{tableaux})\ s_\mu$$

Pieri rule

$$s_{\square}\, s_{(2,1)} = s_{\begin{smallmatrix}\square & & \\ \square & \square & \bullet\end{smallmatrix}} + s_{\begin{smallmatrix}\square & \bullet \\ \square & \square\end{smallmatrix}} + s_{\begin{smallmatrix}\bullet & \\ \square & \\ \square & \square\end{smallmatrix}}$$

Littlewood-Richardson Rule

$$s_\lambda s_\mu = \sum_\nu (\#\ \textit{yamanouchi tableaux})\ s_\nu$$

Grothendieck polynomials: G_λ

Representation theory and algebra

Basis for Λ (indexed by partitions)

Inhomogeneous degree

$$G_{\ydiagram{1,2}} = 2x_1x_2x_3 + x_1x_1x_2 + x_2x_2x_3 + x_1x_1x_3 + x_1x_2x_2 + x_1x_3x_3 + x_2x_3x_3$$

$$-8x_1x_2x_3x_4 + \cdots$$

s_λ is lowest degree term

Grothendieck polynomials: G_λ

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s_λ is lowest degree term

Geometry

K-theory

Schubert representatives for $K^0(Gr_{a,n})$

$G_\lambda \to \mathcal{O}_{\Omega_\lambda}$

Combinatorics of Grothendiecks

Set-valued tableaux

- partition filling using sets
- $\max X < \min Y$ for X below or west of Y

3, 5	9	10	
1, 2	4	6,7,8	11

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Definition

set-valued tableaux generating function

$$G_\lambda = \sum_{\substack{set-valued\ T \\ \text{shape}(T)=\lambda}} (-1)^{|T|-|\lambda|} x^T$$

$$\begin{array}{|c|c} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array} + \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array} + \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c} \hline 3 & \\ \hline 2 & 2 \\ \hline \end{array} + \begin{array}{|c|c} \hline 3 & \\ \hline 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 2 \\ \hline \end{array} + \begin{array}{|c|c} \hline 3 & \\ \hline 1 & 3 \\ \hline \end{array} + \begin{array}{|c|c} \hline 3 & \\ \hline 2 & 3 \\ \hline \end{array}$$

$$- \underbrace{\begin{array}{|c|c} \hline 3 & \\ \hline 12 & 4 \\ \hline \end{array}}_{x_1x_2x_3x_4} - \begin{array}{|c|c} \hline 4 & \\ \hline 12 & 3 \\ \hline \end{array} - \begin{array}{|c|c} \hline 4 & \\ \hline 1 & 23 \\ \hline \end{array} - \begin{array}{|c|c} \hline 24 & \\ \hline 1 & 3 \\ \hline \end{array} - \begin{array}{|c|c} \hline 23 & \\ \hline 1 & 4 \\ \hline \end{array} - \begin{array}{|c|c} \hline 3 & \\ \hline 1 & 24 \\ \hline \end{array} - \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 34 \\ \hline \end{array} - \begin{array}{|c|c} \hline 34 & \\ \hline 1 & 2 \\ \hline \end{array} - \cdots$$

$$+ \begin{array}{|c|c} \hline 4 & \\ \hline 12 & 35 \\ \hline \end{array} + \begin{array}{|c|c} \hline 4 & \\ \hline 1 & 235 \\ \hline \end{array} + \begin{array}{|c|c} \hline 24 & \\ \hline 1 & 35 \\ \hline \end{array} + \begin{array}{|c|c} \hline 23 & \\ \hline 1 & 45 \\ \hline \end{array} + \begin{array}{|c|c} \hline 3 & \\ \hline 1 & 245 \\ \hline \end{array} + \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 345 \\ \hline \end{array} + \begin{array}{|c|c} \hline 34 & \\ \hline 1 & 25 \\ \hline \end{array} +$$

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$$\begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 3 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 2 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 1 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 2 \\ \hline 1 & 2 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 1 & 3 \\ \hline\end{array} + \begin{array}{|c|c|}\cline{1-1} 3 \\ \hline 2 & 3 \\ \hline\end{array}$$

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Kostka Numbers, Pieri rule, LR rule

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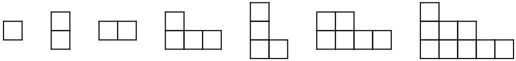
(Dual) k-Schur functions: $\mathfrak{S}_\lambda^{(k)}$ and $s_\lambda^{(k)}$

Symmetric functions

Homogeneous degree

Indexed by $k+1$-cores *(no $k+1$-hooks)*

$k+1=3:$



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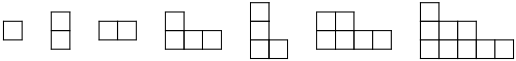

$s_\lambda^{(k)}$ is a basis for $\mathbb{Z}[h_1, \ldots, h_k]$

$\mathfrak{S}_\lambda^{(k)}$ is a basis for $\Lambda / \langle m_\lambda : \lambda_1 > k \rangle$

Macdonald theory

q, t-**Kostka polynomials**

$$H_\lambda = \sum_\mu K_{\lambda\mu}(q,t)\, s_\mu$$

$$H_{2,1,1} = t^2 S_{(4)} + (qt^2 + qt + t)\, S_{(3,1)} + (q^2t^2 + 1)\, S_{(2,2)} + (q^2t + qt + q)\, S_{(2,1,1)} + q^2 S_{(1,1,1,1)}$$

Representation theory

bi-graded characters of S_n-modules

Geometry

Hilbert scheme of points in the plane

Combinatorics

inversion statistics, Dual equivalence graphs,...

Macdonald theory

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Representation theory

bi-graded characters of S_n-modules

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Combinatorics

inversion statistics, Dual equivalence graphs,...

k-Schur theory

$$H_{2,1,1} = t\, s^{(2)}_{(2,2)} + \left(1 + qt^2\right) s^{(2)}_{(2,1,1)} + q\, s^{(2)}_{(1,1,1,1)}$$

(Dual) k-Schur functions

k-Littlewood-Richardson coefficients

$$s_\lambda^{(k)} s_\mu^{(k)} = \sum_\nu c_{\lambda,\mu}^{\nu,k}\, s_\nu^{(k)}$$

- Gromov-Witten invariants
- Quantum Schubert calculus
- WZW model of conformal field theory
- Hecke algebras at roots of unity

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Pieri Rule

(co) homology structure of affine Grassmannian

$\tilde{G}r = SL_n(\mathbb{C}((t)))/SL_n(\mathbb{C}[[t]])$

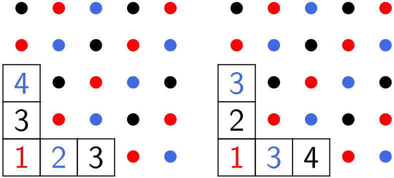
$$s_1^{(k)} s_\lambda^{(k)} = \sum s_\mu^{(k)} \quad \text{and} \quad \mathfrak{S}_1^{(k)} \mathfrak{S}_\lambda^{(k)} = \sum d_\lambda^\mu \mathfrak{S}_\mu^{(k)}$$

Combinatorics of (dual) k-Schurs

***k*-tableaux**

tableaux with $k+1$-core shape

multiplicities have same residue



Combinatorics of (dual) *k*-Schurs

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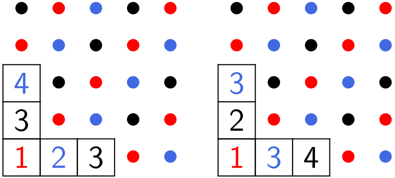
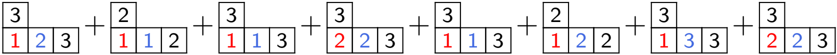


Definition
k-tableaux generating function



$$\mathfrak{S}^{(2)}_{(3,1)} = x_1x_2x_3 + x_1x_1x_2 + x_1x_1x_3 + x_2x_2x_3 + x_1x_1x_3 + x_1x_2x_2 + x_1x_3x_3 + x_2x_2$$

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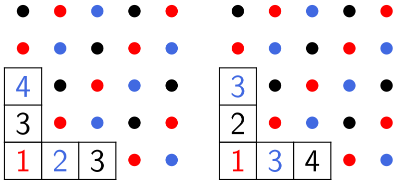

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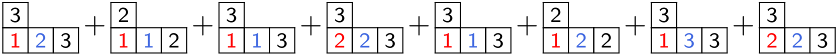

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Definition
$s^{(k)}_\lambda$ is the dual of $\mathfrak{S}^{(k)}_\lambda$ under the Hall-inner product $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

Combinatorics

- ***k*-Kostka Numbers**

  $$h_\lambda = \sum (\# k - \text{tableaux})\, s_\mu^{(k)} = \sum (\# \text{ strong tableaux})\, \mathfrak{S}_\mu^{(k)}$$

Combinatorics

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- **Pieri Rules:** weak/strong order on type-A affine Weyl group

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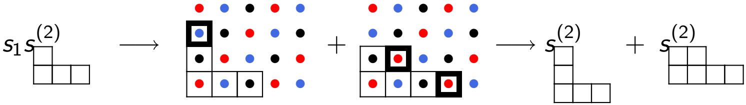
- **Pieri Rules:** weak/strong order on type-A affine Weyl group
  - *k*-**Schurs**

    weak cover = cores differ by one residue



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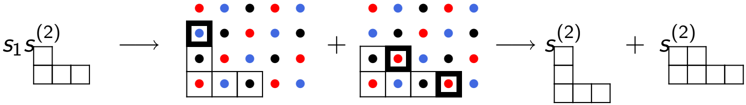
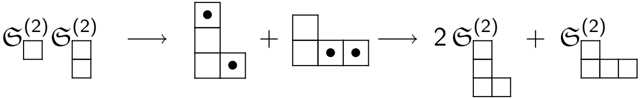


  - **Dual k-Schurs**
    strong cover = number of k-bounded hooks differs by one



Combinatorics

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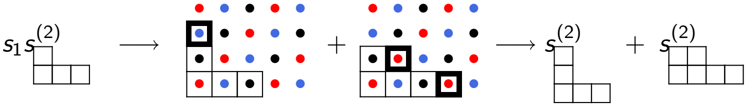

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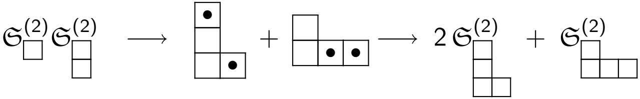

- **Automorphism:** $\omega h_i = e_i$

$$\omega s_\lambda^{(k)} = s_{\lambda'}^{(k)}$$

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Affine Grothendiecks and K-k-Schurs: $G_\lambda^{(k)}$ and $g_\lambda^{(k)}$

- **Symmetric functions**

  Indexed by $k+1$-cores

  Inhomogeneous degree

  $G_\lambda^{(k)}$ is a basis for $\Lambda/\langle m_\lambda : \lambda_1 \geq n\rangle$
  (lowest degree is a dual k-Schur)

  $g_\lambda^{(k)}$ is a basis for $\mathbb{Z}[h_1, \ldots, h_k]$
  (highest degree is a k-Schur)

- **Geometry**

  K-theory of $\tilde{G}r = SL_{k+1}(\mathbb{C}((t)))/SL_{k+1}(\mathbb{C}[[t]])$

  Schubert representatives for $K^0(\tilde{G}r)$ and $K_*(\tilde{G}r)$ classes.

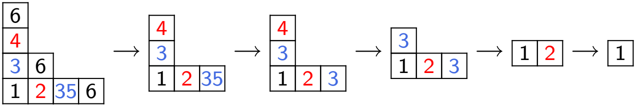
- **Macdonald**

  See next talk

Affine Grothendiecks and K-k-Schurs

Affine set-valued tableaux

- Set-valued tableaux of $k+1$-core shape
- Restricted to $1, 2, \ldots, m$, an m is in corners of some residue



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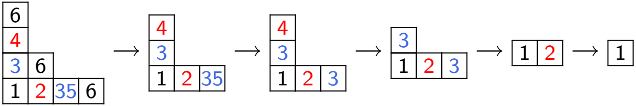
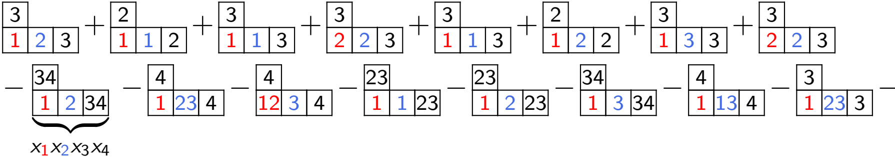


Definition

Affine set-valued tableaux generating function

$$G_\lambda^{(k)} = \sum (-1)^{degree(\lambda) - T} x^T$$



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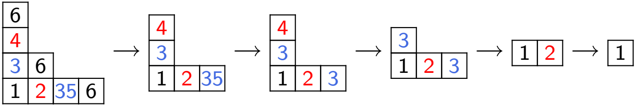

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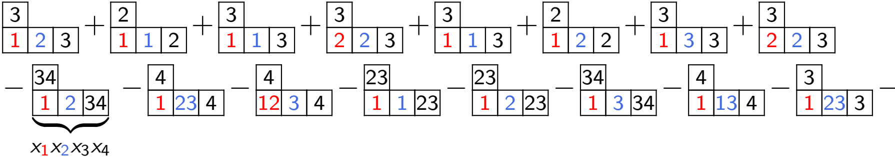


Definition

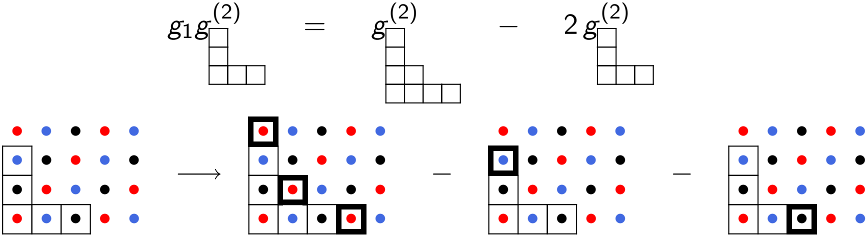
$g_\lambda^{(k)}$ is the dual of $G_\lambda^{(k)}$ under the Hall-inner product

Combinatorics of $g_\lambda^{(k)}$

K-k-**Kostka numbers**

$$h_\lambda = \sum_\mu (\#\ \text{affine set-valued tableaux})\ g_\mu^{(k)}$$

Pieri rule for k-K-Schurs



Automorphism: $\Omega h_r = \sum_{i \geq 1} \binom{r-1}{i-1} e_i$

$$\Omega g_\lambda^{(k)} = g_{\lambda'}^{(k)}$$

Some problems

- Quantum cohomology of Grassmannians
  - LR rule (k-tableaux)
- Quantum K-theory
  - LR rule
  - generating function
- (co)homology of affine Grassmannian
  - LR rule *strong and weak order chains?*
  - involution on strong k-tableaux
  - geometric quantities encoded by structure constants
- affine K-theory
  - Pieri for affine Grothendiecks
  - generating functions for k-K-Schurs
  - quantum meets affine picture?
  - LR-coefficients
- (co)homology and K-theory of affine flags
  - polynomial basis (reduce to k-Schurs/affine Grothendiecks in Grassmannian
  - combinatorial calculation