The shortest path poset of finite Coxeter Groups

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Coxeter Groups

Groups with presentation

$$\langle S \mid (ss')^{m(s,s')} = e, \text{ for all } s, s' \in S \rangle$$

where

- $m(s, s) = 1$
- $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$
- $m(s, s') = \infty$ means that there is no relation between s and s'.

Examples.

- $\mathbb{Z}_2 = \langle s \mid s^2 = 1 \rangle$.

- The dihedral group of order $2m$.
  $I_2(m) = \langle s_1, s_2 \mid (s_1 s_2)^m = (s_2 s_1)^m = s_1^2 = s_2^2 = 1 \rangle$. When $m \geq 3$, this is the group of symmetries of the m-gon.

- The symmetric group.
  $A_{n-1} = S_n = \langle s_1, s_2, \ldots, s_{n-1} \mid (s_i s_j)^{m(s_i, s_j)} \rangle$, where $s_i = (i, i+1)$, $m(s_i, s_{i+1}) = 3$ and otherwise $m(s_i, s_j) = 2$ for $i < j$.

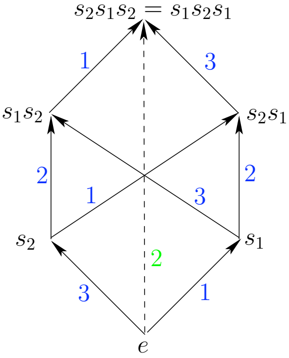
Basic Definitions

- Each $w \in W$ can be expressed as $w = s_1 s_2 \dots s_n$ with $s_i \in S$. If n is minimal, then $s_1 s_2 \dots s_n$ is a reduced expression for w. In this case, we define the length function by $\ell(w) = n$.
- $T(W) = \{wsw^{-1} \mid w \in W, s \in S\}$ is the set of reflections of (W, S).
- Bruhat Order: Let $v, w \in W$. We say that $v \leq w$ if and only if there exist $t_1, \dots, t_k \in T$ so that $vt_1 t_2 \cdots t_k = w$ with $\ell(vt_1) > \ell(v)$ and $\ell(vt_1 \cdots t_i) > \ell(vt_1 \cdots t_{i-1})$ for $i > 1$.
- If W is finite, then there exists a maximal-length word w_0^W; that is, $\ell(w) \leq \ell(w_0^W)$ for all $w \in W$.
- If $|W| < \infty$, then $\ell(w_0^W) = |T(W)|$.

Bruhat Graph

The directed graph (V, E) consisting of $V = W$ and $(u, v) \in E$ if $\ell(u) < \ell(v)$ and there exists $t \in T$ with $ut = v$ is called the Bruhat graph.

For example, consider S_3 with generators $s_1 = (1, 2)$, $s_2 = (2, 3)$, with labeling$1 \to s_1, 2 \to s_1 s_2 s_1, 3 \to s_2$



Reflection Order

A reflection order Is a total order $<_T$ on the reflections of W so that for any dihedral reflection subgroup W' (i.e, W' has two generators, $x, y \in T$) , then either

$$x <_T xyx <_T xyxyx <_T \ldots <_T yxyxy <_T yxy <_T y$$

or

$$y <_T yxy <_T yxyxy <_T \ldots <_T xyxyx <_T xyx <_T x$$

where x and y are the generators of W'.

Complete **cd**-index

Fix a reflection ordering $<_T$. Consider a chain (path) C in the Bruhat graph of $[u, v]$ labeled by reflections, say

$$C = (t_1, t_2, \ldots, t_k)$$

The descent set of C is

$$D(C) = \{i \in [k-1] \mid t_{i+1} <_T t_i\}$$

The complete **cd**-index encodes the descent sets of all the Bruhat paths.

Complete **cd**-index

The encoding is done as follows: Let $\Delta = (t_1, t_2, \ldots, t_k)$ be a path of length k from u to v. Then define $w(\Delta) = x_1 x_2 \cdots x_{k-1}$ where

$$x_i = \begin{cases} \mathbf{a} & \text{if } t_i <_T t_{i+1} \text{(for ascent)} \\ \mathbf{b} & \text{if } t_{i+1} <_T t_i \end{cases}$$

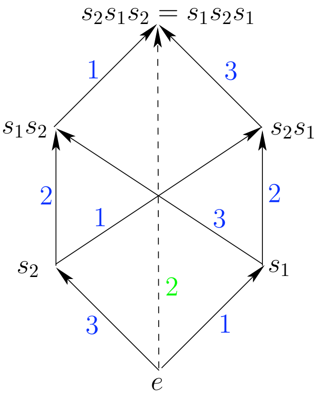
Now consider the polynomial $\sum_\Delta w(\Delta)$. Set

$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$
$$\mathbf{d} = \mathbf{ab} + \mathbf{ba}$$

After the substitution, $\sum_\Delta w(\Delta)$ becomes a polynomial with variables **c** and **d**. This is denoted by $\widetilde{\psi}_{u,v}$, and it is called the complete **cd**-index of $[u, v]$.

Example

Consider S_3 with generators $s_1 = (1,2)$ and $s_2 = (2,3)$, and reflection ordering
$s_1 = (1,2) <_T s_1s_2s_1 = (1,3) <_T s_2 = (2,3)$.



$s_1 <_T s_1s_2s_1 <_T s_2$	
123	a^2
131	ab
313	ba
321	b^2
2	1

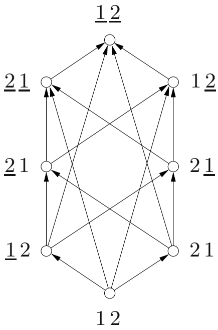
$$\widetilde{\psi}_{e,s_1s_2s_1} = c^2 + 1$$

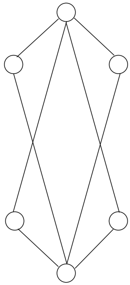
A bigger example

$$\widetilde{\psi}_{12435,53142} = c^5 + 6cdc^2 + 6c^2dc + 3dc^3 + 3c^3d + 7cd^2 + \\ +7d^2c + 6dcd + c^3 + 2dc + 2cd$$

Shortest Path Poset of W

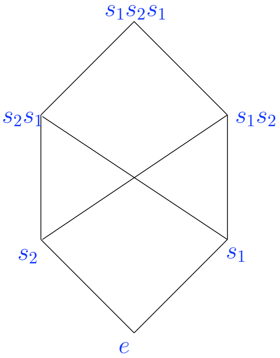
If W is a finite Coxeter group, we can form a poset $SP(W)$ with the shortest paths of W. For example, consider the Bruhat graph of B_2 (signed permutations of two elements)



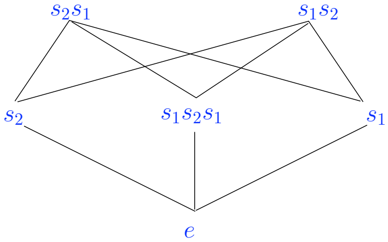


$SP(W)$ is a gra

The absolute length of $w \in W$ is the minimal number of reflections $t_1, \ldots, t_k$ so that $t_1 t_2 \cdots t_k = w$. We write $\ell_T(w) = k$.



Bruhat Order for A_2



Absolute Order for A_2

$SP(A_{n-1})$

How to describe the shortest paths from e to $w_0^{A_{n-1}} = n\ n-1 \dots 2\ 1$?.

Let $r_i = (i \;\; n+1-i)$ and $k = \lfloor \frac{n}{2} \rfloor$. Then

Theorem

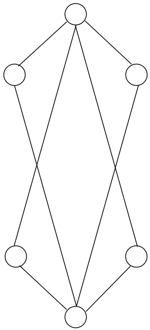
If $t_1 t_2 \cdots t_k = w_0^{A_{n-1}}$ *then*

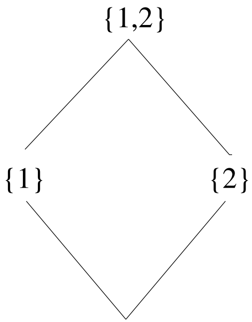
- $\{t_1, t_2, \dots, t_k\} = \{r_1, r_2, \dots, r_k\}$
- $t_i t_j = t_j t_i$ *for all* i, j
- $(t_{\sigma(1)}, t_{\sigma(2)}, \dots, t_{\sigma(k)})$ *is a path in* $B(A_{n-1})$ *for all* $\sigma \in A_{n-1}$.

Corollary

$SP(A_{n-1}) \cong$ Boolean(k), the Boolean poset of rank k (poset of subsets of $\{1, \dots, k\}$ ordered by inclusion).

Example: B_2





$SP(B_2)$ is formed by two copies of *Boolean*(2) that share the smallest and biggest elements.

In general, we have

Theorem

Let W be finite Coxeter group, w_0 the longest element in W, and $\ell_0 = \ell_T(w_0)$. If $t_1 t_2 \cdots t_{\ell_0} = w_0$ then
(a) $t_i t_j = t_j t_i$ for $1 \leq i, j \leq \ell_0$. In particular $t_{\tau(1)} t_{\tau(2)} \cdots t_{\tau(\ell_0)} = w_0$ for all $\tau \in A_{\ell_0 - 1}$.
(b) $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(\ell_0)})$ is a path in the Bruhat graph of W for all $\tau \in A_{\ell_0 - 1}$

Corollary ($SP(W)$)

$SP(W)$ is formed by α_W Boolean posets of rank ℓ_0 (that share the smallest and biggest elements).

W	rank$(SP(W))$	# of Boolean posets
A_n	$\lfloor \frac{n}{2} \rfloor$	1
B_n	n	b_n
D_n	n if n is even; $n-1$ if n is odd	d_n
$I_2(m)$	2 m even; 1 m odd	$\frac{m}{2}$ m even; 1 m odd
F_4	2	1
H_3	3	5
H_4	4	75
E_6	4	3
E_7	7	135
E_8	8	2025

$$b_n = 1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{j!} \prod_{i=0}^{j-1} \binom{n-2i}{2}$$

$$d_n = \frac{1}{\lfloor \frac{m}{2} \rfloor !} \prod_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{n-2i}{2}, \; m = n \text{ if } n \text{ is even. Otherwise } m = n-1.$$

cd-index of *Boolean*(k)

Let $\psi(Boolean(k))$ be the **cd**-index of *Boolean*(k) (that is, the regular **cd**-index of the Eulerian poset *Boolean*(k). Then Ehrenborg and Readdy show that

$$\psi(Boolean(1)) = 1$$

$$\psi(Boolean(k)) = \psi(Boolean(k-1)) \cdot \mathbf{c} + G(\psi(Boolean(k-1))$$

G is the derivation (derivation means $G(xy) = xG(y) + G(x)y$) $G(\mathbf{c}) = \mathbf{d}$ and $G(\mathbf{d}) = \mathbf{cd}$.
For example

$$\psi(Boolean(2)) = \mathbf{c}$$

$$\psi(Boolean(3)) = \mathbf{c}^2 + \mathbf{d}$$

$$\psi(Boolean(4)) = \mathbf{c}^3 + 2(\mathbf{cd} + \mathbf{dc})$$

Theorem

The lowest-degree terms of $\widetilde{\psi}_{e,w_0}$ *are given by* $\alpha_W \psi(Boolean(\ell_T(w_0)))$ *for some* $\alpha_W \in \mathbb{Z}$.

Corollary

The lowest-degree terms of $\widetilde{\psi}_{e,w_0}$ are minimized (component-wise) by $\psi(Boolean(\ell_0))$.

This corollary is true for the lowest degree terms of $\psi_{e,v}$ if $[c^{\ell_0 - 1}] = 1$, where $[c^k]$ is denotes the coefficient of c^k in $\psi_{e,v}$.

Conjecture: Corollary holds for $\widetilde{\psi}_{u,v}$.