Edge Labellings of Partially Ordered Sets

by

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Abstract

It is well known that if a finite graded lattice of rank n is supersolvable, then it has an EL-labelling where the labels along any maximal chain form a permutation of $\{1, 2, \ldots, n\}$. We call such a labelling an S_n EL-labelling and we show that a finite graded lattice of rank n is supersolvable if and only if it has such a labelling. This result can be used to show that a graded lattice is supersolvable if and only if it has a maximal chain of left modular elements.

We next study finite graded bounded posets that have S_n EL-labellings and describe a type A 0-Hecke algebra action on their maximal chains. This action is local and the resulting representation of these Hecke algebras is closely related to the flag h-vector. We show that finite graded lattices of rank n, in particular, have such an action if and only if they have an S_n EL-labelling.

Our next goal is to extend these equivalences to lattices that need not be graded and, furthermore, to bounded posets that need not be lattices. In joint work with Hugh Thomas, we define left modularity in this setting, as well as a natural extension of S_n EL-labellings, known as interpolating labellings. We also suitably extend the definition of lattice supersolvability to arbitrary bounded graded posets. We show that these extended definitions preserve the appropriate equivalences.

Finally, we move to the study of P-partitions. Here, edges are labelled as either "strict" or "weak" depending on an underlying labelling of the elements of the poset. A well-known conjecture of R. Stanley states that the quasisymmetric generating function for P-partitions is symmetric if and only if P is isomorphic to a Schur labelled skew shape poset. In characterizing these skew shape posets in terms of their local structure, C. Malvenuto made significant progress on this conjecture. We generalize the definition of P-partitions by letting the set of strict edges be arbitrary. Using cylindric diagrams, we extend Stanley's conjecture and Malvenuto's characterization to this setting. We conclude by proving both conjectures for large classes of posets.

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Chapter 1

Introduction

The study of edge labellings of partially ordered sets (posets) has its beginnings in the work of Richard Stanley [35]. An edge labelling is nothing more than a map from the edges of the Hasse diagram of a poset to the positive integers. Despite this apparent simplicity, showing that a poset admits a particular type of edge labelling can yield important information about its combinatorial and topological properties. Perhaps the most successful such type of edge labelling is the class of EL-labellings, defined by Anders Björner in [5] and motivated by examples from [35], [36] and [37]. If a poset has an EL-labelling, then we know that its associated simplicial complex is shellable and hence Cohen-Macaulay. Furthermore, EL-labellings yield combinatorial interpretations for important invariants of a poset, such as its Möbius function, flag f-vector and flag h-vector. We will be interested in a particular subclass of EL-labellings known as "S_n EL-labellings." Their definition has additional combinatorial appeal in that S_n EL-labellings of a poset are EL-labellings where the labels along any maximal chain of the poset form a permutation of the set $\{1, 2, \ldots, n\}$.

In Chapter 2, we investigate the connection between S_n EL-labellings and a classical lattice property. Supersolvable lattices were introduced by Stanley in [36] as a generalization of distributive lattices. Examples of supersolvable lattices that are not necessarily distributive include the lattice of subgroups of a supersolvable group (hence the terminology) and the lattice of partitions of a set. Using different terminology, Stanley showed that supersolvable lattices admit S_n EL-labellings. As our first main result, we will show that the converse is also true and so a lattice is supersolvable if and only if it has an S_n EL-labelling. Before proving this result, we give background, definitions and examples to fully explain and motivate the above concepts. As an application, we give an example of how this characterization of lattice supersolvability in terms of S_n EL-labellings can be used to give short alternative proofs of otherwise difficult supersolvability results.

One of the tools used in Chapter 2 is a naturally defined action on the maximal chains of a poset with an S_n EL-labelling. This action has a number of desirable properties. The first is that it is a local action, in that it only changes a maximal chain in at most one place. Local actions on the maximal chains of a poset have received considerable recent attention: see, for example, [16], [17], [34], [39] and [40]. Furthermore, as explained in proper detail in Section 2.5, the generators of our action satisfy essentially the same axioms as the generators of a particular Hecke algebra of type A. We thus refer to the action as a "local $\mathcal{H}_n(0)$ action." Since we are working with posets with S_n EL-labellings, our definition applies to all supersolvable posets. The special case of this action for a distributive lattice is discussed in [11].

We begin Chapter 3 by giving an introduction to quasisymmetric functions, as there are two quasisymmetric functions related to our action. The first, defined by Richard Ehrenborg [13], depends only on the flag f-vector, or flag h-vector, of the poset in question. The second is what is known as the "characteristic" of the character of the defining representation of our local $\mathcal{H}_n(0)$ action, as defined in [12] and [20]. Perhaps the most significant property of our action is that its two quasisymmetric functions are essentially equal. Thus, we follow [34] and [39] in calling our action a "good" $\mathcal{H}_n(0)$ action. The primary aim of Chapter 3 is to determine what posets have good $\mathcal{H}_n(0)$ actions. Our main result of the chapter is that a lattice has a good $\mathcal{H}_n(0)$ action if and only if it has an S_n EL-labelling. In fact, our result holds for a more general class of posets, and we discuss some examples and counterexamples in Section 3.4. The work of Chapters 2 and 3, which also appears in a more concise form as [27], brings together three seemingly different concepts from the theory of ordered sets: supersolvability, edge labellings and actions on maximal chains.

As a consequence of our first two main results, we have given two new characterizations of lattice supersolvability. In Section 4.1, we see that our results can be used to give a third new characterization. Stanley showed in [36] that a supersolvable lattice has a maximal chain of left modular elements. Applying a result of Larry Liu [21], Hugh Thomas observed that we can now deduce that a graded lattice is supersolvable if and only if it has such a left modular maximal chain. This is a considerable strengthening of a result of Stanley [36] which says that these two properties are equivalent in a semimodular lattice.

In Chapter 4, which is joint work with Thomas, we seek to extend our equivalences to lattices which need not be graded and, furthermore, to posets which need not be lattices. We define a natural extension of left modularity and we introduce "interpolating labellings," a natural extension of S_n EL-labellings. We then show the relevant desired result: an arbitrary bounded poset has a left modular maximal chain if and only if it has an interpolating labelling. It is now natural to ask for an appropriate extended definition of supersolvability. We conclude Chapter 4 by giving such a definition and by showing that, for a bounded graded poset, supersolvability, possessing a left modular maximal chain and having an S_n EL-labelling are all equivalent properties.

Chapter 5 discusses the topic of P-partitions, which also has its beginnings in [35]. While the material of this final chapter is essentially separate from that of the earlier chapters, the work is similar in nature. First, quasisymmetric functions play a prominent role and we show an explicit connection with Ehrenborg's quasisymmetric function. Secondly, P-partitions are also concerned with a certain type of labelling of the edges of a poset. Indeed, we begin with a (vertex) labelled poset, and this labelling determines a designation of the edges of the poset as either "strict" or "weak." A P-partition can be defined to be a map from the elements of the poset to the positive integers that is order-preserving and that is strictly order preserving along strict edges. Our object of study is a certain quasisymmetric generating function for all P-partitions of a given poset. In fact, Ira Gessel's original definition of quasisymmetric functions [14] was motivated by this particular generating function. We wish to determine the conditions on our poset which makes this generating function symmetric, and a well-known conjecture of Stanley [35, page 81] states that the generating function is symmetric if and only if our poset is isomorphic to what is known as a "Schur labelled skew shape" poset. For these skew shape posets, the generating functions are precisely the skew Schur functions, which are known to be symmetric. The conjecture has been shown to be true by Stanley for posets with no strict edges and has been verified by John Stembridge for all posets with at most seven elements. Given that one would have to determine information on the global structure of our labelled poset from the symmetry of its P-partition generating function, it is hardly surprising that Stanley's conjecture remains open. However, Claudia Malvenuto [25] provided a far less daunting formulation of the conjecture by characterizing these skew shape posets in terms of their local structure. Sections 5.1 and 5.2 give an exposition of this background material.

From the definition of P-partitions, we see that they depend only on the designation of strict and weak edges, and not directly on the underlying labelling of the poset. This suggests that we might take a poset with an arbitrary designation of strict and weak edges and study the resulting natural generalization of P-partitions. Using the idea of cylindric diagrams [30] (see also [2], [15]), we are led to an extension of Stanley's conjecture for these generalized P-partitions. Our first result of this chapter is to extend Malvenuto's reformulation to this setting. Finally, we conclude by proving Stanley's conjecture and our generalized version for large classes of posets.

Our goal throughout is to give a self-contained exposition that would be accessible to a graduate student with some familiarity with posets and basic combinatorics. While we have aimed to give a clear formulation and proof of our main results, we have also striven to display the results in a meaningful context by sharing intuition and providing any necessary or especially relevant background.

Chapter 2

EL-labellings and supersolvability

2.1 Preliminaries

We will write the cardinality of a set S as $|S|$ or $\#S$, and $\mathbb{P}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$ will denote the set of positive integers, integers, rational numbers and complex numbers, respectively. For any positive integer n, we will write $[n]$ for the set $\{1, 2, \ldots, n\}$, and $\{a_1, a_2, \ldots, a_j\}_<$ will signify that $a_1 < a_2 < \cdots < a_j$. Throughout, we let s_i denote the permutation which transposes i and $i+1$, and composition of permutations will be from right to left.

All of our posets will be finite. If $x < y$ in a poset P and there does not exist z in P such that $x < z < y$, then we say that y *covers* x and we write $x \lessdot y$. We will write $x \leq\!\!\cdot\; y$ to mean that y either covers or equals x. Let $\mathcal{E}(P) = \{(s,t) : s \lessdot t \text{ in } P\}$, the set of edges of the Hasse diagram of P, and let $\mathcal{M}(P)$ denote the set of maximal chains of P. (For undefined poset terminology, see Appendix A.) We will say that P is *bounded* if it contains a unique minimal and a unique maximal element, denoted $\hat{0}$ and $\hat{1}$ respectively.

We now define one of our main objects of study: an *edge labelling* γ of a poset P is simply a function $\gamma : \mathcal{E}(P) \rightarrow \mathbb{Z}$. If $\mathfrak{m} : s = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = t$ is a maximal chain of the interval $[s,t]$, then we write $\gamma(\mathfrak{m}) = (\gamma(x_0, x_1), \gamma(x_1, x_2), \ldots, \gamma(x_{k-1}, x_k))$. The chain $\mathfrak{m}$ is said to be *increasing* if $\gamma(x_0, x_1) \leq \gamma(x_1, x_2) \leq \cdots \leq \gamma(x_{k-1}, x_k)$. We define the *descent set* $D(\pi)$ of an integer sequence $\pi = (a_1, a_2, \ldots, a_j)$ by

$$D(\pi) = \{i : a_i > a_{i+1}\}.$$

We define the set of inversions $INV(\pi)$ of π by

$$INV(\pi) = \{(a_j, a_i) : i < j, a_i > a_j\}.$$

Suppose a poset P has an edge labelling γ and $\mathfrak{m}$ is a maximal chain of P. It is natural to define the descent set of $\mathfrak{m}$ to be the descent set of $\gamma(\mathfrak{m})$ and the set of inversions of $\mathfrak{m}$ to be the set of inversions of $\gamma(\mathfrak{m})$. We let $\leq_L$ denote lexicographic order on finite integer sequences, defined by:

$$(a_1, a_2, \ldots, a_j) \leq_L (b_1, b_2, \ldots, b_k)$$

if either

(i) $j \leq k$ and $a_i = b_i$ for $i = 1, 2, \ldots, j$, or

(ii) $a_i < b_i$ for the least i satisfying $a_i \neq b_i$.

Finally, suppose P is a bounded graded poset of rank n. Let rk denote the rank function of P, so $\mathrm{rk}(\hat{0}) = 0$ and $\mathrm{rk}(\hat{1}) = n$. If $x \leq y$ in P, let $\mathrm{rk}(x, y)$ denote $\mathrm{rk}(y) - \mathrm{rk}(x)$.

2.2 EL-labellings

The idea of studying edge labellings of posets goes back to [35]. An important milestone was Anders Björner's introduction of EL-labellings.

Definition 2.2.1. Let P be a finite poset. An edge labelling $\gamma : \mathcal{E}(P) \to \mathbb{Z}$ is called an *EL-labelling* if the following two conditions are satisfied:

(i) Every interval $[s, t]$ has exactly one increasing maximal chain $\mathfrak{m}$.

(ii) Any other maximal chain $\mathfrak{m}'$ of $[s, t]$ satisfies $\gamma(\mathfrak{m}') >_L \gamma(\mathfrak{m})$.

This concept originates in [5] with motivating examples coming from [35] and [36], which appear here as Examples 2.2.3 and 2.3.5 respectively. For the case when P need not be graded, see [8, 9]. A poset P with an EL-labelling is said to be *edge-wise lexicographically shellable* or *EL-shellable*. While this definition of EL-labellings applies to any finite poset, for the remainder of this chapter we will only be concerned with EL-labellings of bounded graded posets.

Example 2.2.2. Consider the poset B_n, the set of subsets of $[n]$. If y covers x in B_n then $y - x = \{i\}$ for some $i \in [n]$ and we set $\gamma(x, y) = i$. This defines an EL-labelling for B_n.

Example 2.2.3. Any finite distributive lattice is EL-shellable. Let L be a finite distributive lattice of rank n. By the Fundamental Theorem of Finite Distributive Lattices [4, p. 59, Thm. 3], that is equivalent to saying that $L = J(P)$, the lattice of order ideals of some n-element poset P. Let $\omega : P \to [n]$ be a linear extension of P, i.e., any bijection labelling the vertices of P that is order-preserving (if $a \leq b$ in P then $\omega(a) \leq \omega(b)$). This labelling of the vertices of P defines a labelling of the edges of $J(P)$ as follows: if y covers x in $J(P)$, then the order ideal corresponding to y is obtained from the order ideal corresponding to x by adding a single element, labelled by i, say. Then we set $\gamma(x, y) = i$. This gives us an EL-labelling for $L = J(P)$. Figure 2-1 shows a labelled poset and its lattice of order ideals with the appropriate edge labelling.

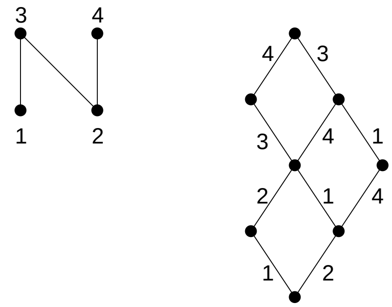


Figure 2-1: An EL-labelling of a distributive lattice

The ubiquity and usefulness of EL-labellings arises from the fact that if P is EL-shellable, then P is shellable and hence Cohen-Macaulay. Further information on these concepts can be found in [5] and the highly recommended survey article [6]. EL-labellings also give a simple combinatorial interpretation of some fundamental invariants of a poset, as we now explain.

Let P be a bounded graded poset of rank n and let $S \subseteq [n-1]$. We define the *S-rank selected subposet* P_S of P by

$$P_S = \{x \in P : \mathrm{rk}(x) \in S\} \cup \left\{\hat{0}, \hat{1}\right\}$$

Notice that P_S is graded of rank $|S|+1$. Let $\alpha_P(S)$ denote the number of maximal chains of P_S. In other words,

$$\alpha_P(S) = \#\left\{\hat{0} < x_1 < x_2 < \cdots < x_{|S|} < \hat{1} : \left\{\mathrm{rk}(x_1), \mathrm{rk}(x_2), \ldots, \mathrm{rk}(x_{|S|})\right\} = S\right\}.$$

The function $\alpha_P : 2^{[n-1]} \to \mathbb{Z}$ is known as the *flag f-vector* of P. It contains equivalent information to the *flag h-vector* β_P of P defined by

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T). \tag{2.1}$$

By Inclusion-Exclusion, this is equivalent to the definition

$$\alpha_P(S) = \sum_{T \subseteq S} \beta_P(T). \tag{2.2}$$

The following result appears, in increasing order of generality, in [35], [36], [37] and [5].

Theorem 2.2.4. *Let P be a finite bounded graded poset of rank n with an EL-labelling γ. Then, for all $S \subseteq [n-1]$, $\beta_P(S)$ is equal to the number of maximal chains of P with descent set S. In particular, $\beta_P(S) \geq 0$.*

The proof below is modeled on Stanley's proof of Theorem 3.13.2 in [38].

Proof. To each maximal chain $\mathfrak{c} : \hat{0} = x_0 < x_1 < \cdots < x_{|S|+1} = \hat{1}$ of P_S we wish to associate a maximal chain $\mathfrak{m}$ of P with descent set contained in S. Since γ is an

EL-labelling, the interval $[x_i, x_{i+1}]$ in P has exactly one increasing maximal chain $\mathfrak{m}_i$. Thinking of chains in terms of their elements, we let

$$\mathfrak{m} = \mathfrak{m}_0 \cup \mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_{|S|}.$$

It is clear that this gives a bijection between maximal chains of P_S and maximal chains of P with descent set contained in S. Therefore, $\alpha_P(S)$ equals the number of maximal chains of P with descent set contained in S. Applying Inclusion-Exclusion, we conclude that $\beta_P(S)$ is the number of maximal chains of P with descent set equal to S. □

Note. We only used property (i) of EL-labellings in this proof. Edge labellings with this property are known as *R-labellings*.

Part of Stanley's motivation for studying the flag h-vector was to obtain information about the sign of the Möbius function of bounded graded posets. If $x \leq y$ in a poset P, then the Möbius function $\mu_P(x, y)$ is given by

$$\mu_P(x, y) = c_0 - c_1 + c_2 - c_3 + \cdots,$$

where c_i is the number of chains $x = x_0 < x_1 < \cdots < x_i = y$ of length i between x and y. If P is graded and $x < y$, letting $S = \{\mathrm{rk}(x)+1, \mathrm{rk}(x)+2, \ldots, \mathrm{rk}(y)-1\}$, we see that

$$\begin{aligned} \mu_P(x, y) &= \sum_{T \subseteq S} (-1)^{|T|+1} \alpha_{[x,y]}(T) \\ &= (-1)^{|S|+1} \sum_{T \subseteq S} (-1)^{|S|-|T|} \alpha_{[x,y]}(T) \\ &= (-1)^{\mathrm{rk}(x,y)} \beta_{[x,y]}(S). \end{aligned}$$

Since every interval of an EL-shellable poset inherits an EL-labelling, we get the following combinatorial interpretation of the Möbius function of an EL-shellable poset.

Corollary 2.2.5. *Let P be a finite bounded graded poset with an EL-labelling γ. When $x < y$ in P, $(-1)^{\mathrm{rk}(x,y)} \mu_P(x, y)$ is equal to the number of maximal chains $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = y$ of $[x, y]$ satisfying*

$$\gamma(x_0, x_1) > \gamma(x_1, x_2) > \cdots > \gamma(x_{k-1}, x_k).$$

In particular,

$$(-1)^{\mathrm{rk}(x,y)} \mu_P(x, y) \geq 0.$$

2.3 S_n EL-labellings

The reader may have noticed that the two examples of EL-labellings given in the previous section had the property that the labels along any maximal chain, when

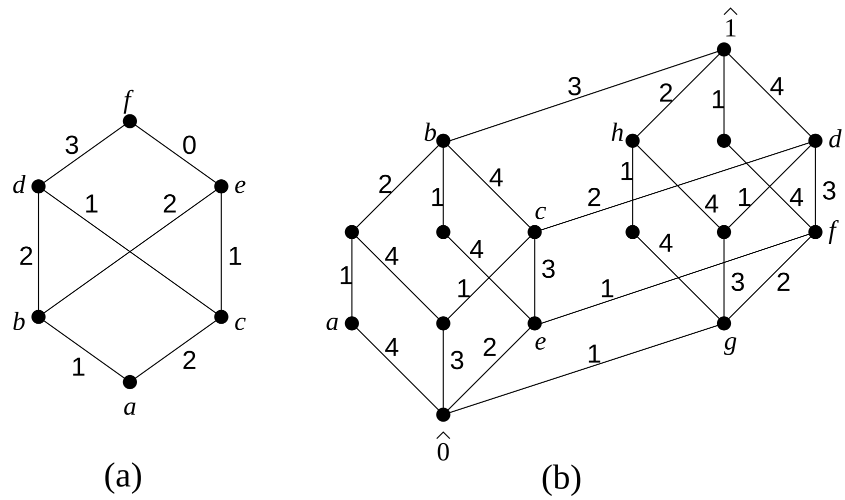


Figure 2-2: Two posets that are EL-shellable but not snellable

read from bottom to top, formed a permutation. EL-labellings with this property will be a major focus of our work.

Definition 2.3.1. An EL-labelling γ of a bounded graded poset P of rank n is said to be an S_n *EL-labelling* if, for every maximal chain $\mathfrak{m} : \hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ of P, the map sending i to $\gamma(x_{i-1}, x_i)$ is a permutation of $[n]$. In other words, $\gamma(\mathfrak{m})$ is an element of the permutation group S_n written in the usual way.

If a poset P has an S_n EL-labelling, or *snelling* for short, then it is said to be S_n *EL-shellable*, or *snellable* for short. Note that the second condition in the definition of an EL-labelling is redundant in this case.

Example 2.3.2. The posets shown in Figure 2-2 are seen to be EL-shellable. However, it can be shown that neither of them is snellable. Notice that the second poset, unlike the first, is a lattice. It appears, together with this EL-labelling, in [32]. To build some intuition for these labellings, it is worthwhile to see why these posets fail to have snellings.

(a) We wish to give this poset a snelling γ. Without loss of generality, we can let the maximal chain $a \lessdot b \lessdot d \lessdot f$ be the increasing maximal chain. Since the chain $a \lessdot b \lessdot e \lessdot f$ must be labelled by a permutation, we must have that $\{\gamma(b, e), \gamma(e, f)\} = \{2, 3\}$. In fact, since $b \lessdot d \lessdot f$ is the increasing chain in $[b, \hat{1}]$, we must have $\gamma(b, e) = 3$ and $\gamma(e, f) = 2$. Similarly, $\gamma(a, c) = 2$ and $\gamma(c, d) = 1$. But then the maximal chain $a \lessdot c \lessdot e \lessdot f$ will not be labelled by a permutation.

Note. This explanation highlights a useful fact about snellings: in intervals of rank 2 that are "diamond" shaped (i.e. have two elements of rank 1), opposite edges must receive the same label.

(b) Again, suppose we wish to give this lattice a snelling γ. Since the interval $[a, \hat{1}]$ must have an increasing chain, we must have $\gamma(b, \hat{1})$ equaling 3 or 4. By the note above applied to the interval $[c, \hat{1}]$, we must also have $\gamma(c, d) \in \{3, 4\}$. Similarly, $\gamma(e, f) \in \{3, 4\}$ and $\gamma(\hat{0}, g) \in \{3, 4\}$. But then it is impossible for the interval $[\hat{0}, h]$ to have an increasing chain.

Before proceeding with examples, we present the following uniqueness result for snellings.

Lemma 2.3.3. *Let γ and δ be two S_n EL-labellings of a bounded graded poset P. If γ and δ have the same increasing maximal chain, then γ and δ coincide.*

Proof. Let $\mathfrak{m} : \hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ be a maximal chain with lexicographically minimal γ labelling among those chains for which γ and δ disagree. Since $\mathfrak{m}$ is not the increasing chain from $\hat{0}$ to $\hat{1}$ of γ, we can find an i such that $\gamma(x_{i-1}, x_i) > \gamma(x_i, x_{i+1})$. Since the interval $[x_{i-1}, x_{i+1}]$ must have an increasing chain, there exists a maximal chain $\mathfrak{m}'$ of P which differs from $\mathfrak{m}$ only a rank i and which has labels from γ which are increasing in this interval. Then $\gamma(\mathfrak{m}') <_L \gamma(\mathfrak{m})$, so γ and δ agree on $\mathfrak{m}'$. Since the labelling of $\mathfrak{m}'$ determines the labelling of $\mathfrak{m}$, γ and δ must also agree on $\mathfrak{m}$. Thus they agree everywhere. □

Example 2.3.4. We wish to look more closely at the snelling for distributive lattices described in Example 2.2.3. Given a distributive lattice $L = J(P)$, we would like to be able to give it a snelling without referring to P. Pick any maximal chain $\mathfrak{m} : \hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ of $J(P)$ and let $\gamma(\mathfrak{m}) = (1, 2, \ldots, n)$. This determines a linear extension of P which in turn determines a snelling for $J(P)$ as in Example 2.2.3. By Lemma 2.3.3, this is the only snelling of $J(P)$ that satisfies $\gamma(\mathfrak{m}) = (1, 2, \ldots, n)$.

Let $y \lessdot z$ be any edge of $J(P)$ and suppose $z - y = \{i\}$ as order ideals of P. Then we have that

$$i = \min \{j : x_j \vee y \geq z\}$$

since joins in $J(P)$ are just set unions and the order ideal corresponding to x_j is $\{1, 2, \ldots, j\}$. This gives us a way to specify γ without referring to P. Explicitly, for $y \lessdot z$ in $L = J(P)$, setting

$$\gamma(y, z) = \min \{j : x_j \vee y \geq z\}$$

gives us the unique snelling of $J(P)$ that has $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ as its increasing maximal chain.

Example 2.3.5. The set of supersolvable lattices is our final example and is also the example most relevant to our work. These lattices will be the subject of the next section.

2.4 Supersolvable lattices

The following definition first appeared in [36].

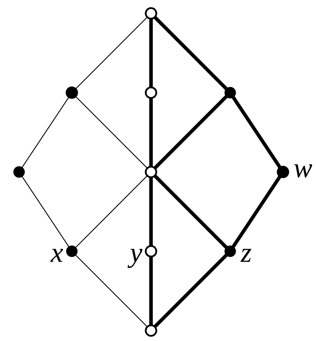


Figure 2-3: A supersolvable lattice that is not distributive

Definition 2.4.1. A finite lattice L is said to be *supersolvable* if it contains a maximal chain, called an *M-chain* of L, which together with any other chain in L generates a distributive sublattice.

Supersolvable lattices can be thought of as a generalization of distributive lattices. Indeed, this introduction of supersolvable lattices allowed Stanley to extend results for distributive lattices from [35] to this larger class of lattices. One of these extensions appears below as Theorem 2.4.7.

Example 2.4.2. Distributive lattices are supersolvable. This is a direct consequence of the following lemma.

Lemma 2.4.3. *Any sublattice of a distributive lattice is distributive.*

Proof. Let L be a distributive sublattice. Recall that this is equivalent to the statement that

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{2.3}$$

for all x, y, z in L. Let K be any sublattice of L and let x and y be elements of K. By definition of K, $x \vee_L y$ and $x \wedge_L y$ must be in K and must equal $x \vee_K y$ and $x \wedge_K y$ respectively. Therefore, (2.3) also holds in K for all elements x, y and z of K. □

Example 2.4.4. Figure 2-3 shows a supersolvable lattice with the M-chain indicated by open dots. Since x, y and z do not satisfy Equation (2.3), this lattice is not distributive. The sublattice generated by the M-chain and the chain $z < w$ is shown with bold edges, and is seen to be isomorphic to the distributive lattice of Figure 2-1.

Example 2.4.5. This example explains the terminology "supersolvable lattice."

Definition 2.4.6. A finite group G is *supersolvable* if it has a sequence of subgroups

$$\{e\} \subset G_1 \subset G_2 \subset \cdots \subset G_m = G \tag{2.4}$$

such that each G_i is normal in G and G_{i+1}/G_i is cyclic of prime order.

As shown in [36, Example 2.5], the lattice of subgroups $L(G)$ of a supersolvable group G is a supersolvable lattice, with M-chain given by (2.4). In fact, as shown in [5, Theorem 3.3], $L(G)$ is supersolvable if and only if G is supersolvable.

The following result is of central importance to the main questions addressed in this thesis. It was first shown in [36] and a self-contained proof in a more general setting is given in Volkmar Welker's survey article [44]. It will also follow from our observations in Section 4.1. However, we have already built sufficient background to give an elementary proof.

Theorem 2.4.7. *Let L be finite supersolvable lattice of rank n. Then L has an S_n EL-labelling.*

Proof. Let $M : \hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ denote the M-chain of L. Given any chain $\mathfrak{c}$ of L, we know that M and $\mathfrak{c}$ together generate a distributive lattice. The main idea is to give this distributive lattice a snelling γ as in Example 2.3.4 so that $\gamma(M) = (1, 2, \ldots, n)$. By varying $\mathfrak{c}$, we can label every edge of L. Our proof of the validity of this approach divides into three parts. First, we must show that this assigns a unique label to each edge of L. Next, we will show that each maximal chain of L is labelled by a permutation. Finally, we will show that each interval has exactly one increasing chain.

We showed in Example 2.3.4 that, in the sublattice generated by M and any $\mathfrak{c}$, the label on an edge $y \lessdot z$ is given by

$$\gamma(y, z) = \min \{j : x_j \vee y \geq z\}. \tag{2.5}$$

Now the expression $x_j \vee y \geq z$ is either true in every such sublattice containing y and z or is false in every such sublattice. Therefore, each edge $y \lessdot z$ is assigned a unique label.

It is clear that all the labels are in the set $\{1, 2, \ldots, n\}$. Suppose that a maximal chain $\mathfrak{m}$ of L contains two edges with the same label. Then these two edges must have received the same label in the sublattice K generated by M and $\mathfrak{m}$. But this is impossible, since the labelling of K is a snelling. We conclude that every maximal chain of L is labelled by a permutation.

It is clear from our construction that each interval will have at least one increasing chain. Let $[y, z]$ be an interval of minimal rank with two increasing chains,

$$y = u_0 \lessdot u_1 \lessdot \cdots u_k = z \quad \text{and} \quad y = w_0 \lessdot w_1 \lessdot \cdots w_k = z.$$

Since every maximal chain of L is labelled by a permutation and since both chains are increasing, we must have $\gamma(u_{i-1}, u_i) = \gamma(w_{i-1}, w_i) = l_i$ for $i = 1, 2, \ldots, k$. Also, by the minimality of $[y, z]$, we must have $u_1 \neq w_1$ and by the definition (2.5) of the labelling, we must have $x_{l_1} \vee y \geq u_1$ and $x_{l_1} \vee y \geq w_1$. Therefore, $x_{l_1} \vee y \geq u_1 \vee w_1$. It follows that for every edge $y' \lessdot z'$ in the interval $[u_1, u_1 \vee w_1]$, we have $x_{l_1} \vee y' \geq z'$ and so $\gamma(y', z') \leq l_1$. Furthermore, since $\gamma(u_0, u_1) = l_1$, we have $\gamma(y', z') < l_1$. But the edge $y' \lessdot z'$ is in the interval $[y, z]$ and all the edges of $[y, z]$ must be in the set $\{l_1, l_2, \ldots, l_k\}_<$. This gives a contradiction and so every interval has exactly one increasing maximal chain. □

This brings us to the first question addressed in this program of research.

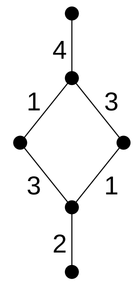


Figure 2-4: The action on the maximal chains

Question 2.4.8. What other graded lattices, apart from supersolvable lattices, have S_n EL-labellings?

We are now in a position to state our first main result.

Theorem 2.4.9. *A finite graded lattice of rank n has an S_n EL-labelling if and only if it is supersolvable.*

We will prove Theorem 2.4.9 in Section 2.6 but first, we wish to study a general property of snellable lattices.

2.5 $\mathcal{H}_n(0)$ actions

Let P be a bounded graded poset of rank n with a snelling γ, and let $\mathfrak{m}$ be a maximal chain of P. Suppose $\mathfrak{m}$ has a descent at i. Since P is snellable, there exists exactly one chain $\mathfrak{m}' : \hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{i-1} \lessdot x_i' \lessdot x_{i+1} \lessdot \cdots \lessdot x_n = \hat{1}$ differing only from $\mathfrak{m}$ at rank i and having no descent at i. This suggests the following definition of functions $U_i : \mathcal{M}(P) \to \mathcal{M}(P)$:

Definition 2.5.1. Let P be a finite bounded graded poset of rank n with an S_n EL-labelling. Let $\mathfrak{m}$ be a maximal chain of P. We define $U_1, U_2, \ldots U_{n-1} : \mathcal{M}(P) \to \mathcal{M}(P)$ by $U_i(\mathfrak{m}) = \mathfrak{m}'$, where $\mathfrak{m}'$ is the unique maximal chain of P differing only from $\mathfrak{m}$ at possibly rank i and having no descent at i.

Under this definition, we see that the descent set of a maximal chain $\mathfrak{m}$ of P can also be defined to be the set

$$\{i \in [n-1] : U_i(\mathfrak{m}) \neq \mathfrak{m}\}. \tag{2.6}$$

This definition will be used in Chapter 3 for posets P where no snelling is defined.

Observe that when $\mathfrak{m}' \neq \mathfrak{m}$, $\gamma(\mathfrak{m}')$ is the same as $\gamma(\mathfrak{m})$ except that the ith and $(i+1)$st elements have been switched. In other words, $\gamma(\mathfrak{m}') = \gamma(\mathfrak{m})s_i$. Figure 2-4 shows an example for the case $n = 4$. Let $\mathfrak{m}$ be the maximal chain to the left. It has a descent at 2 and therefore $\mathfrak{m}' = U_2(\mathfrak{m}) \neq \mathfrak{m}$. The labels of $\mathfrak{m}'$ are forced by the fact that $\mathfrak{m}'$ does not have a descent at 2. We have that $\gamma(\mathfrak{m}') = \gamma(\mathfrak{m})s_2$.

We see that the action of $U_1, U_2, \ldots, U_{n-1}$ has the following properties:

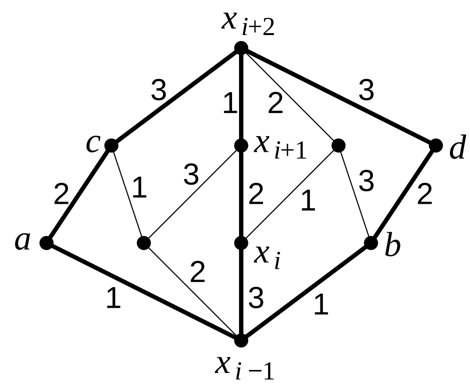


Figure 2-5: $U_iU_{i+1}U_i = U_{i+1}U_iU_{i+1}$

1. It is a *local* action, i.e., $U_i(\mathfrak{m})$ agrees with $\mathfrak{m}$ except possibly at the ith rank. Local actions on the maximal chains of a poset have been studied, for example, in [16], [17], [34], [39] and [40].

2. ${U_i}^2 = U_i$ for $i = 1, 2, \ldots, n-1$. This differs from most of the local actions in the aforementioned papers which were symmetric group actions and so satisfied ${U_i}^2 = 1$.

3. $U_iU_j = U_jU_i$ if $|i - j| \geq 2$.

4. $U_iU_{i+1}U_i = U_{i+1}U_iU_{i+1}$ for $i = 1, 2, \ldots, n-2$. Showing this requires a routine check of each of the six possible relative orderings of $\gamma(x_{i-1}, x_i)$, $\gamma(x_i, x_{i+1})$ and $\gamma(x_{i+1}, x_{i+2})$. The only non-trivial and interesting case is when

$$\gamma(x_{i-1}, x_i) > \gamma(x_i, x_{i+1}) > \gamma(x_{i+1}, x_{i+2}).$$

   The "proof by picture" for this case is given in Figure 2-5 where, without loss of generality, we assume that the labels appearing in the interval $[x_{i-1}, x_{i+2}]$ are simply 1,2 and 3. The original chain is shown in the center, with the action of $U_iU_{i+1}U_i$ drawn toward the left and the action of $U_{i+1}U_iU_{i+1}$ drawn toward the right. Since $[x_{i-1}, x_{i+2}]$ has only one increasing chain, we must have $a = b$ and $c = d$, giving that $U_iU_{i+1}U_i = U_{i+1}U_iU_{i+1}$.

Now we compare this to the definition of the Hecke algebra of S_n and of the corresponding 0-Hecke algebra.

Definition 2.5.2. Let $\mathbb{C}(q)$ be the field of rational functions in the variable q. The *Hecke algebra* $\mathcal{H}_n$ of type A_{n-1} is the $\mathbb{C}(q)$-algebra generated by $T_1, T_2, \ldots, T_{n-1}$ with relations:

(i) ${T_i}^2 = (q-1)T_i + q$ for $i = 1, 2, \ldots, n-1$.

(ii) $T_iT_j = T_jT_i$ if $|i - j| \geq 2$.

(iii) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ for $i = 1, 2, \ldots, n-2$.

For more information, see [19, §7.4] and [31]. Setting $q = 0$ suggests the following definition of the 0-Hecke algebra $\mathcal{H}_n(0)$ as studied in [11], [12], [20] and [29].

Definition 2.5.3. The *0-Hecke algebra* $\mathcal{H}_n(0)$ of type A_{n-1} is the $\mathbb{C}$-algebra generated by $T_1, T_2, \ldots, T_{n-1}$ with relations:

(i) $T_i^2 = -T_i$ for $i = 1, 2, \ldots, n-1$.

(ii) $T_iT_j = T_jT_i$ if $|i - j| \geq 2$.

(iii) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ for $i = 1, 2, \ldots, n-2$.

We can extend the action of $U_1, U_2, \ldots, U_{n-1}$ on $\mathcal{M}(P)$ to a linear action on $\mathbb{C}\mathcal{M}(P)$, the complex vector space with basis $\mathcal{M}(P)$. If we set $T_i = -U_i$ then $U_1, U_2, \ldots, U_{n-1}$ generate the 0-Hecke algebra and so we can now refer to our action on the maximal chains of P as a *local* $\mathcal{H}_n(0)$ *action.* In [11, §3.9], Gérard Duchamp, Florent Hivert and Jean-Yves Thibon describe the special case of this action on distributive lattices. They work in the language of linear extensions of a poset Q which, as we have seen, correspond to snellings of $J(Q)$.

2.6 Snellable implies supersolvable

Our main aim for this section is to prove Theorem 2.4.9.

Let L be a finite graded lattice of rank n. By Theorem 2.4.7, we know that if L is supersolvable, then L is snellable. Now we suppose that L has a snelling γ and we wish to prove that L is supersolvable. We let M denote the unique maximal chain of L labelled by the identity permutation. Taking M to be our candidate M-chain, we let $L_{\mathfrak{c}}$ denote the sublattice of L generated by M and any other chain $\mathfrak{c}$ of L.

If $\mathfrak{c}$ is a chain in L that isn't maximal, then we can extend it to a maximal chain $\mathfrak{m}$ in at least one way. Then $L_{\mathfrak{c}}$ is a sublattice of $L_{\mathfrak{m}}$, so by Lemma 2.4.3, $L_{\mathfrak{c}}$ is distributive if $L_{\mathfrak{m}}$ is distributive. Therefore, it suffices to show that $L_{\mathfrak{m}}$ is distributive for all maximal chains $\mathfrak{m}$. Our approach will be to define two new posets, $Q_{\mathfrak{m}}$ and $P_{\gamma(\mathfrak{m})}$, and to show that

$$L_{\mathfrak{m}} = Q_{\mathfrak{m}} \cong J(P_{\gamma(\mathfrak{m})}).$$

Much of what we will say will be true for posets that need not be lattices. With this in mind, let P be a finite bounded graded poset of rank n with a snelling γ. We have seen that if $U_i(\mathfrak{m})$ differs from $\mathfrak{m}$, then $U_i(\mathfrak{m})$ has one less inversion than $\mathfrak{m}$ and that $\gamma(U_i(\mathfrak{m})) = \gamma(\mathfrak{m})s_i$. It follows that if $\mathfrak{m}$ has r inversions then we can find a sequence $U_{i_1}, U_{i_2}, \ldots, U_{i_r}$ such that $U_{i_1}U_{i_2}\cdots U_{i_r}(\mathfrak{m}) = M$. We define $\mathcal{M}_{\mathfrak{m}}$, a subset of $\mathcal{M}(P)$, as follows:

$$\mathcal{M}_{\mathfrak{m}} = \{\mathfrak{m}' \in \mathcal{M}(P) : \exists\ i_1, i_2, \ldots, i_r \text{ such that } \mathfrak{m}' = U_{i_1}U_{i_2}\cdots U_{i_r}(\mathfrak{m})\}$$

We label the elements of $\mathcal{M}_{\mathfrak{m}}$ as they are labelled in P. We define $Q_{\mathfrak{m}}$ to be the subposet of P with elements

$$\{u \in L : u \in \mathfrak{m}' \text{ for some } \mathfrak{m}' \in \mathcal{M}_{\mathfrak{m}}\}$$

and with a partial order inherited from P. $Q_{\mathfrak{m}}$ can be thought of as the closure of $\mathfrak{m}$ in P under the operations $U_1, U_2, \ldots, U_{n-1}$. We should note that it is not obvious that every maximal chain of $Q_{\mathfrak{m}}$ is in $\mathcal{M}_{\mathfrak{m}}$. We wish to obtain a clear picture of the structure of $Q_{\mathfrak{m}}$. We will now state and prove a series of facts about $Q_{\mathfrak{m}}$, each one of which brings us a step closer to proving Theorem 2.4.9.

Fact 1. *Let $\mathfrak{m}'$ and $\mathfrak{m}''$ be distinct elements of $\mathcal{M}_{\mathfrak{m}}$. Then $\gamma(\mathfrak{m}') \neq \gamma(\mathfrak{m}'')$.*

Proof. Suppose that $\gamma(\mathfrak{m}') = \gamma(\mathfrak{m}'')$. Let $U_{i_1}, U_{i_2}, \ldots, U_{i_l}$ and $U_{j_1}, U_{j_2}, \ldots, U_{j_l}$ be sequences of minimal length such that $\mathfrak{m}' = U_{i_1}U_{i_2}\cdots U_{i_l}(\mathfrak{m})$ and $\mathfrak{m}'' = U_{j_1}U_{j_2}\cdots U_{j_l}(\mathfrak{m})$. Now $\gamma(U_i(\mathfrak{m}_1)) = \gamma(\mathfrak{m}_1)s_i$ whenever $\mathfrak{m}_1$ is a maximal chain of P with $U_i(\mathfrak{m}_1) \neq \mathfrak{m}_1$. By the minimality of the sequences above, it follows that

$$\gamma(\mathfrak{m}') = \gamma(\mathfrak{m}'') = \gamma(\mathfrak{m})s_{i_1}s_{i_{l-1}}\cdots s_{i_1} = \gamma(\mathfrak{m})s_{j_l}s_{j_{l-1}}\cdots s_{j_1}.$$

Thus $s_{i_1}s_{i_2}\cdots s_{i_l}$ and $s_{j_1}s_{j_2}\cdots s_{j_l}$ are expressions for $\gamma(\mathfrak{m}')^{-1}\gamma(\mathfrak{m})$ in terms of a minimal number of adjacent transpositions. We say that $s_{i_1}s_{i_2}\cdots s_{i_l}$ and $s_{j_1}s_{j_2}\cdots s_{j_l}$ are *reduced expressions*, or *reduced decompositions*, for $\gamma(\mathfrak{m}')^{-1}\gamma(\mathfrak{m})$. Tits' Word Theorem (see [19, §8.1], [43]) for S_n states that any two reduced expressions for a given permutation can be obtained from each other by a sequence of *braid moves* (i.e. replace $s_is_{i+1}s_i$ by $s_{i+1}s_is_{i+1}$ or vice versa or replace s_is_j by s_js_i if $|i-j| \geq 2$). In particular, $s_{i_1}s_{i_2}\cdots s_{i_l}$ can be obtained from $s_{j_1}s_{j_2}\cdots s_{j_l}$ by a sequence of braid moves. But by Properties 3 and 4 of the U_i action, $U_{i_1}U_{i_2}\cdots U_{i_l}(\mathfrak{m})$ is invariant under braid moves and hence

$$U_{i_1}U_{i_2}\cdots U_{i_l}(\mathfrak{m}) = U_{j_1}U_{j_2}\cdots U_{j_l}(\mathfrak{m}).$$

We conclude that $\mathfrak{m}' = \mathfrak{m}''$, which is a contradiction. Therefore, $\gamma(\mathfrak{m}') \neq \gamma(\mathfrak{m}'')$. □

Fact 2. *Let $u \in Q_{\mathfrak{m}}$. Then there is a unique chain $\mathfrak{m}_u \in \mathcal{M}_{\mathfrak{m}}$ that has increasing labels between $\hat{0}$ and u and between u and $\hat{1}$.*

Proof. Choose any $\mathfrak{m}' \in \mathcal{M}_{\mathfrak{m}}$ such that $u \in \mathfrak{m}'$. Suppose u has rank i in P. Apply $U_1, U_2, \ldots, U_{i-1}, U_{i+1}, \ldots, U_{n-1}$ repeatedly to $\mathfrak{m}'$ to obtain $\mathfrak{m}_u$. The chain $\mathfrak{m}_u$ is unique in $\mathcal{M}_{\mathfrak{m}}$ because it is unique in P. □

Fact 3. *To each point u of $Q_{\mathfrak{m}}$ we can associate the subset Γ_u of $[n]$ consisting of the labels on any maximal chain of $[\hat{0}, u]$ in P. Then any two distinct points of $Q_{\mathfrak{m}}$ correspond to distinct subsets of $[n]$.*

Proof. Let u, v be distinct elements of $Q_{\mathfrak{m}}$ and suppose $\Gamma_u = \Gamma_v$. Then $\gamma(\mathfrak{m}_u) = \gamma(\mathfrak{m}_v)$, contradicting Fact 1. □

An important tool for the remainder of the proof will be the weak order on permutations of $[n]$.

Definition 2.6.1. Let v, w be permutations of $[n]$. We say that $v \leq_R w$ if there exist $i_1, i_2, \ldots, i_r$ such that $v = ws_{i_r}s_{i_{r-1}}\cdots s_{i_1}$ and $ws_{i_r}\cdots s_{i_{k+1}}s_{i_k}$ has one less inversion than $ws_{i_r}\cdots s_{i_{k+1}}$ for $k = 1, 2, \ldots, r$.

It is known (see, for example, [7, Prop. 2.5]) that $v \leq_R w$ if and only if $INV(v) \subseteq INV(w)$.

Fact 4. *The labels on the elements of $\mathcal{M}_{\mathfrak{m}}$ consist of all those permutations ω satisfying $\omega \leq_R \gamma(\mathfrak{m})$, each occurring exactly once.*

Proof. Compare the definitions of $\mathcal{M}_{\mathfrak{m}}$ and $\leq_R$. We see that if $\mathfrak{m}' \in \mathcal{M}_{\mathfrak{m}}$ then $\gamma(\mathfrak{m}') \leq_R \gamma(\mathfrak{m})$ and if $\omega \leq_R \gamma(\mathfrak{m})$ then there exists $\mathfrak{m}' \in \mathcal{M}_{\mathfrak{m}}$ satisfying $\gamma(\mathfrak{m}') = \omega$. The fact that ω occurs only once follows from Fact 1. □

Fact 5. *Let $u, v \in Q_{\mathfrak{m}}$. It is clear that if $u \leq v$ then $\Gamma_u \subseteq \Gamma_v$. Suppose $\Gamma_u \subseteq \Gamma_v$ for some elements u, v of $Q_{\mathfrak{m}}$. Then $u \leq v$.*

Proof. Construct a permutation ω as follows:

- Let $\omega(1), \omega(2), \ldots, \omega(|\Gamma_u|)$ be the elements of Γ_u taken in increasing order.
- Let $\omega(|\Gamma_u|+1), \ldots, \omega(|\Gamma_v|)$ be the elements of $\Gamma_v - \Gamma_u$ taken in increasing order.
- Let $\omega(|\Gamma_v|+1), \ldots, \omega(n)$ be the elements of $[n] - \Gamma_v$ taken in increasing order.

Then, since $u, v \in Q_{\mathfrak{m}}$, we have that

$$INV(\omega) \subseteq INV(\gamma(\mathfrak{m}_u)) \cup INV(\gamma(\mathfrak{m}_v)) \subseteq INV(\gamma(\mathfrak{m}))$$

and so $\omega \leq_R \gamma(\mathfrak{m})$. Let $\mathfrak{m}_{u,v}$ be the element of $\mathcal{M}_{\mathfrak{m}}$ from Fact 4 satisfying $\gamma(\mathfrak{m}_{u,v}) = \omega$. By Fact 3, u and v are both elements of $\mathfrak{m}_{u,v}$. We conclude that $u \leq v$ in $Q_{\mathfrak{m}}$. □

In other words, this fact combined with Fact 3 tells us that the map $\varphi : Q_{\mathfrak{m}} \to B_n$ defined by $\varphi(u) = \Gamma_u$ is a poset embedding.

We can now exhibit a poset $P_{\gamma(\mathfrak{m})}$ such that $Q_{\mathfrak{m}} \cong J(P_{\gamma(\mathfrak{m})})$. For any $\omega \in S_n$, we can construct P_ω, a poset on $[n]$ with relation $\leq$ defined by $i \leq j$ if and only if $(i,j) \notin INV(\omega)$. For example, if $\omega = 2413$ we get the poset on the left in Figure 2-1.

Proposition 2.6.2. *Let P be a finite bounded graded poset with a snelling γ and let $\mathfrak{m}$ be a maximal chain of P. The map $\phi : Q_{\mathfrak{m}} \to J(P_{\gamma(\mathfrak{m})})$ defined by $\phi(u) = \Gamma_u$ is a poset isomorphism.*

Proof. Suppose Γ_u has size k.

$$\begin{aligned} u \in Q_{\mathfrak{m}} &\Leftrightarrow \Gamma_u = \{\omega(1), \omega(2), \ldots, \omega(k)\} \text{ for some } \omega \leq_R \gamma(\mathfrak{m}) \\ &\Leftrightarrow \Gamma_u = \{\omega(1), \omega(2), \ldots, \omega(k)\} \text{ for some } \omega \text{ satisfying} \\ &\qquad INV(\omega) \subseteq INV(\gamma(\mathfrak{m})) \\ &\Leftrightarrow \Gamma_u \text{ is an order ideal of } P_{\gamma(\mathfrak{m})} \\ &\Leftrightarrow \Gamma_u \in J(P_{\gamma(\mathfrak{m})}). \end{aligned}$$

Therefore, ϕ is a well-defined bijection. If u and v are elements of $Q_{\mathfrak{m}}$, by Fact 5,

$$u \leq v \text{ in } Q_{\mathfrak{m}} \Leftrightarrow \Gamma_u \subseteq \Gamma_v \Leftrightarrow \Gamma_u \leq \Gamma_v \text{ in } J(P_{\gamma(\mathfrak{m})}) \tag{2.7}$$

as required. □

In particular, it follows that the structure of $Q_{\mathfrak{m}}$ depends only on $\gamma(\mathfrak{m})$ and not even on the underlying poset P. It is now time to revert to the case when $P = L$ is a finite graded lattice of rank n with a snelling γ. Of course, everything we have shown above for P will be true in the case $P = L$.

Proof of Theorem 2.4.9. We now know that $Q_{\mathfrak{m}}$ is a distributive lattice. We next show that $Q_{\mathfrak{m}}$ is a sublattice of L.

Let $u, v \in Q_{\mathfrak{m}}$ with corresponding subsets Γ_u and Γ_v, respectively. Recall that $u \vee_L v$ denotes the join of u and v in L and that $u \vee_{Q_{\mathfrak{m}}} v$ denotes the join of u and v in $Q_{\mathfrak{m}}$, which we now know is a lattice. In L we have that

$$u \vee_{Q_{\mathfrak{m}}} v \;\geq\; u \vee_L v$$

since $Q_{\mathfrak{m}}$ is a subposet of L. But by (2.7),

$$\mathrm{rk}(u \vee_{Q_{\mathfrak{m}}} v) = |\Gamma_u \cup \Gamma_v| \leq \mathrm{rk}(u \vee_L v)$$

since there are maximal chains of $[\hat{0}, u \vee_L v]$ going through u and others going through v. Thus,

$$u \vee_{Q_{\mathfrak{m}}} v \;=\; u \vee_L v.$$

Similarly,

$$u \wedge_{Q_{\mathfrak{m}}} v \;=\; u \wedge_L v.$$

Therefore, $Q_{\mathfrak{m}}$ is a distributive sublattice of L. Furthermore, $L_{\mathfrak{m}}$ is a sublattice of $Q_{\mathfrak{m}}$ since $L_{\mathfrak{m}}$ is a sublattice of L and $Q_{\mathfrak{m}}$ contains $\mathfrak{m}$ and M. By Lemma 2.4.3, we conclude that $L_{\mathfrak{m}}$ is also distributive and hence L is supersolvable. □

The astute reader will notice that, while we have shown that L is supersolvable and that $L_{\mathfrak{m}} \subseteq Q_{\mathfrak{m}}$, we have not fulfilled our promise to show that $L_{\mathfrak{m}} = Q_{\mathfrak{m}}$. However, this follows from the following lemma.

Lemma 2.6.3. *For each element $\mathfrak{m}'$ of $\mathcal{M}_{\mathfrak{m}}$, we have $Q_{\mathfrak{m}'} = L_{\mathfrak{m}'}$.*

Proof. Let $\mathfrak{m}'$ be an element of $\mathcal{M}_{\mathfrak{m}}$ such that $\gamma(\mathfrak{m}')$ has l inversions. The proof is by induction on l with the result being trivially true for $l = 0$. Since we know that $L_{\mathfrak{m}'} \subseteq Q_{\mathfrak{m}'} \subseteq Q_{\mathfrak{m}}$, it suffices to restrict our attention to $Q_{\mathfrak{m}}$. In this setting, we can label the elements of $Q_{\mathfrak{m}}$ by their corresponding subsets of $[n]$. By (2.7), join and meet in $Q_{\mathfrak{m}}$ are just set union and set intersection, respectively.

Referring to Figure 2-6, suppose $\mathfrak{m}'$ is the vertical chain. Suppose that $|T| = i - 1$ and $a > b$ so that $\mathfrak{m}'$ has a descent at rank i. Now

$$T + \{b\} = ((T + \{a, b\}) \cap (\{1, 2, \dots, a - 1\})) \cup T$$

and $\{1, 2, \dots, a-1\} \in M$. Therefore, $T + \{b\} \in L_{\mathfrak{m}'}$ and so we get that $L_{U_i(\mathfrak{m}')} \subseteq L_{\mathfrak{m}'}$

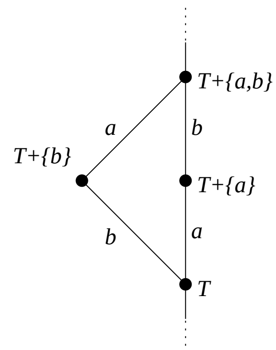


Figure 2-6: Showing $U_i(\mathfrak{m}')$ is in $L_{\mathfrak{m}}$

as sets. Suppose the descents of $\mathfrak{m}'$ are at ranks $i_1, i_2, \ldots, i_k$. Then, as sets,

$$\begin{aligned} Q_{\mathfrak{m}'} &= Q_{U_{i_1}(\mathfrak{m}')} \cup Q_{U_{i_2}(\mathfrak{m}')} \cup \cdots \cup Q_{U_{i_k}(\mathfrak{m}')} \cup \mathfrak{m}' \\ &= L_{U_{i_1}(\mathfrak{m}')} \cup L_{U_{i_2}(\mathfrak{m}')} \cup \cdots \cup L_{U_{i_k}(\mathfrak{m}')} \cup \mathfrak{m}' \quad \text{by induction} \\ &\subseteq L_{\mathfrak{m}'}. \end{aligned}$$

□

Example 2.6.4. Let Π_{n+1} denote the lattice of partitions of the set $[n+1]$ into blocks, where we order the partitions by *refinement*: if μ and ν are partitions of $[n+1]$ we say that $\mu \leq \nu$ if every block of μ is contained in some block of ν. Equivalently, ν covers μ in Π_{n+1} if and only if ν is obtained from μ by merging two blocks of μ. Therefore, if μ has k blocks, then $\mathrm{rk}(\mu) = n+1-k$ and Π_{n+1} has rank n. Π_{n+1} is shown to be supersolvable in [36] and hence can be given a snelling γ as in Theorem 2.4.7. We can choose the M-chain to be the maximal chain consisting of the bottom element and those partitions of $[n+1]$ whose only non-singleton block is $[i]$ where $2 \leq i \leq n+1$. In the literature, this snelling γ is often defined in the following form, which can be shown to be equivalent. If ν is obtained from μ be merging the blocks B and B', then we set

$$\gamma(\mu, \nu) = \max\{\min B, \min B'\} - 1.$$

A *non-crossing partition* of $[n+1]$ is a partition with the property that if some block B contains i and k and some block B' contains j and l with $i < j < k < l$ then $B = B'$. Again, we can order the set of non-crossing partitions of $[n+1]$ by refinement and we denote the resulting poset by NC_{n+1}. This poset, which can be shown to be a lattice, has many nice properties and has been studied extensively. More information on NC_{n+1} can be found in Rodica Simion's survey article [33] and the references given there. Since NC_{n+1} is a subposet of Π_{n+1}, we can consider γ restricted to the edges of NC_{n+1}. It was observed by Björner and Paul Edelman in [5] that this gives an EL-labelling for NC_{n+1} and we can easily see that this EL-labelling is, in fact, a snelling of NC_{n+1}. As a folklore result, NC_{n+1} has been known to be supersolvable and a proof using chain decompositions can be found in [17] or [18]. Theorem 2.4.9

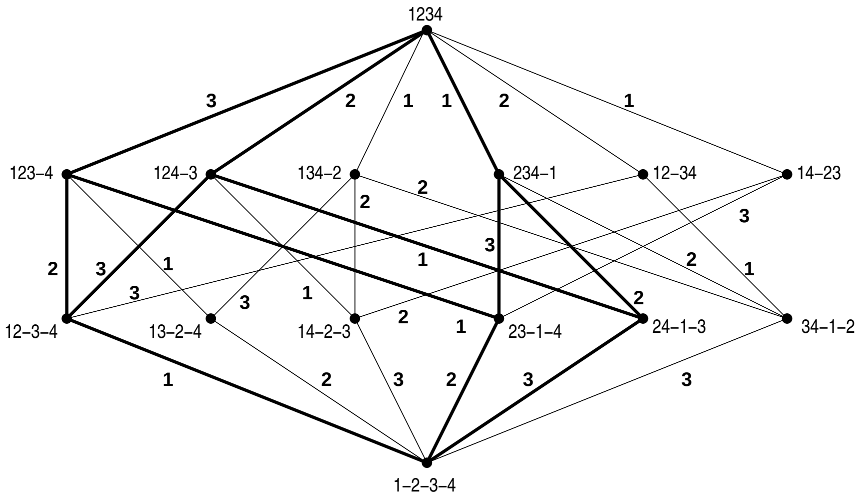


Figure 2-7: NC_4 with snelling

now gives a new proof of the supersolvability of NC_{n+1}. Figure 2-7 shows NC_4 with $L_{\mathfrak{m}} = Q_{\mathfrak{m}}$ highlighted for when $\mathfrak{m}$ is the maximal chain $\hat{0} < \text{24-1-3} < \text{234-1} < \hat{1}$. In this case, $P_{\gamma(\mathfrak{m})}$ is just 3 incomparable elements and so $J(P_{\gamma(\mathfrak{m})}) = B_3 \cong Q_{\mathfrak{m}}$.

Example 2.6.5. In [3], Riccardo Biagioli and Frédéric Chapoton define a class of lattices on forests of leaf-labelled binary trees. As one of the main results, the authors prove that intervals in these lattices are supersolvable by giving these intervals explicit snellings and then applying Theorem 2.4.9.

Chapter 3

Good $\mathcal{H}_n(0)$ actions

Let P be a finite bounded graded poset of rank n. We saw in Section 2.5 that there is a natural way to define a local $\mathcal{H}_n(0)$ action on the maximal chains of P. Intriguingly, there is a very close connection between this action and the flag f-vector and flag h-vector as discussed in Section 2.2. We will begin with the necessary background.

3.1 Quasisymmetric functions

If f is a formal power series that can be expanded in term of some basis $\{B_\alpha\}$ indexed by α, then we will use the standard notation $[B_\beta]\, f$ to denote the coefficient of B_β in this expansion of f.

Richard Ehrenborg in [13, Def. 3] suggested looking at the formal power series (in the variables $x = (x_1, x_2, \ldots)$)

$$F_P(x) = \sum_{\hat{0}=t_0 \leq t_1 \leq \cdots \leq t_{k-1} < t_k = \hat{1}} x_1^{\mathrm{rk}(t_0,t_1)} x_2^{\mathrm{rk}(t_1,t_2)} \cdots x_k^{\mathrm{rk}(t_{k-1},t_k)}, \tag{3.1}$$

where the sum is over all multichains from $\hat{0}$ to $\hat{1}$ such that $\hat{1}$ occurs exactly once. Note that omitting the requirement that $\hat{1}$ occurs only once would cause the coefficients to be infinite. We will refer to $F_P(x)$ as "Ehrenborg's flag function." It is easy to see that the series $F_P(x)$ is homogeneous of degree n. Also notice that it is a *quasisymmetric function*:

Definition 3.1.1. A *quasisymmetric function* in the variables $x_1, x_2, \ldots$, say with rational coefficients, is a formal power series $f = f(x) \in \mathbb{Q}[[x_1, x_2, \ldots]]$ of bounded degree such that for every sequence $n_1, n_2, \ldots, n_m \in \mathbb{P}$ of exponents,

$$\left[x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_m}^{n_m}\right] f = \left[x_{j_1}^{n_1} x_{j_2}^{n_2} \cdots x_{j_m}^{n_m}\right] f$$

whenever $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_m$.

Notice that we get the definition of a symmetric function when we change the condition that the sequences $i_1, i_2, \ldots, i_m$ and $j_1, j_2, \ldots, j_m$ be strictly increasing to the weaker condition that each sequence consists of distinct elements. As an example, the formal power series

$$f(x) = \sum_{1 \le i < j} x_i^2 x_j$$

is a quasisymmetric function but is not a symmetric function. While they appeared implicitly in earlier work, the definition of quasisymmetric functions is due to Ira Gessel [14]. Gessel's definition was motivated by a generating function for P-partitions that will be a focal point of our studies in Chapter 5.

If $\tau = (\tau_1, \ldots, \tau_k)$ is a composition of n, then we define the *monomial quasisymmetric function* $M_{\tau,n}$ by

$$M_{\tau,n} = \sum_{1 \le i_1 < \cdots < i_k} x_{i_1}^{\tau_1} \cdots x_{i_k}^{\tau_k}. \tag{3.2}$$

It is clear that the set $\{M_\tau\}$, where τ ranges over all compositions of n, forms a basis for the vector space of quasisymmetric functions of degree n. As we know, compositions of n are in bijection with subsets of $[n-1]$. Letting $S = \{\tau_1, \tau_1 + \tau_2, \ldots, \tau_1 + \cdots + \tau_{k-1}\}$, we will often denote the quasisymmetric function of Equation (3.2) by $M_{S,n}$. There is an alternative basis that is more important for our purposes. Given $S \subseteq [n-1]$, define the *fundamental quasisymmetric function* $L_{S,n}$ by

$$L_{S,n}(x) = \sum_{\substack{1 \le i_1 \le i_2 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

We see that

$$L_{S,n} = \sum_{S \subseteq T \subseteq [n-1]} M_{T,n}$$

and so by Inclusion-Exclusion,

$$M_{S,n} = \sum_{S \subseteq T \subseteq [n-1]} (-1)^{|T-S|} L_{T,n}.$$

Hence the set $\{L_{S,n}\}$, where S ranges over all subsets of $[n-1]$, forms a basis for the vector space of quasisymmetric functions of degree n. In fact, it is not hard to show (see [41, Exercise 7.93]) that the vector space of all quasisymmetric functions is closed under multiplication and hence can be referred to as the *algebra of quasisymmetric functions over* $\mathbb{Q}$. We now wish to express $F_P(x)$ in terms of our two bases.

Lemma 3.1.2. *Let P be a finite graded bounded poset of rank n with flag f-vector α_P and flag h-vector β_P. Then*

$$F_P(x) = \sum_{S \subseteq [n-1]} \alpha_P(S) M_{S,n}(x) = \sum_{S \subseteq [n-1]} \beta_P(S) L_{S,n}(x). \tag{3.3}$$

Proof. The first equality is true by inspection of the definitions of $F_P(x)$, $\alpha_P(S)$ and

$M_{S,n}(x)$. We have

$$\begin{aligned} F_P(x) &= \sum_{S\subseteq[n-1]} \alpha_P(S) M_{S,n}(x) \\ &= \sum_{S\subseteq[n-1]} \alpha_P(S) \sum_{S\subseteq T\subseteq[n-1]} (-1)^{|T-S|} L_{T,n}(x) \\ &= \sum_{T\subseteq[n-1]} \left(\sum_{S\subseteq T} (-1)^{|T-S|} \alpha_P(S) \right) L_{T,n}(x) \\ &= \sum_{T\subseteq[n-1]} \beta_P(T) L_{T,n}(x) \end{aligned}$$

by Equation (2.1). □

The case when F_P is a symmetric function is considered in [34] and [39] and we wish, in a sense, to extend this to the case when F_P is a quasisymmetric function. In our references to symmetric functions, we follow the notation of [23]. The usual involution ω on symmetric functions given by $\omega(s_\lambda) = s_{\lambda'}$ can be extended to the set of quasisymmetric functions by the definition $\omega(L_{S,n}) = L_{[n-1]-S,n}$. [1] As in [41, Exercise 7.94], where this extended definition appears, we leave it as an exercise for the reader familiar with quasisymmetric functions to check that it restricts to the ring of symmetric functions to give the usual ω. This involution ω also appears in [13], where it is defined in terms of the monomial quasisymmetric functions.

3.2 Good actions

We now introduce some representation theory related to our local $\mathcal{H}_n(0)$ action. In the symmetric function case, certain classes of posets P have been found to have the property that

$$F_P(x) = \mathrm{ch}(\psi) \quad \text{or} \quad \omega F_P(x) = \mathrm{ch}(\psi)$$

where ψ denotes the character of some local symmetric group action and where $\mathrm{ch}(\psi)$ denotes its Frobenius characteristic as defined in [23, §I.7]. In extending these concepts to the $\mathcal{H}_n(0)$ case, we follow the definitions in [12] and [20]. We will use the same notation as in Section 2.5: P is a bounded graded poset of rank n with a snelling, the functions $U_1, U_2, \ldots, U_{n-1}$ give us a local $\mathcal{H}_n(0)$ action on $\mathcal{M}(P)$ and $T_1, T_2, \ldots, T_{n-1}$ are the generators of the 0-Hecke algebra, with $T_i = -U_i$. The representation theory of $\mathcal{H}_n(0)$ is studied by Norton in [29]. There are known to be 2^{n-1} irreducible representations, all of dimension 1. Since $T_i^2 = -T_i$, the irreducible representations are obtained by sending a set of generators to -1 and its complement to 0. We will label these representations by subsets S of $[n-1]$, and then the irreducible representation

[1] This is slightly different from the involution ω on quasisymmetric functions from [25] and [26].

ψ_S of $\mathcal{H}_n(0)$ is defined by

$$\psi_S(T_i) = \begin{cases} -1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

Therefore,

$$\psi_S(U_i) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

Hence the character of ψ_S, denoted by χ_S, is given by

$$\chi_S(U_{i_1}U_{i_2}\cdots U_{i_k}) = \begin{cases} 1 & \text{if } i_j \in S \text{ for } j = 1, 2, \ldots, k \ , \\ 0 & \text{otherwise.} \end{cases}$$

We define its *characteristic* by

$$\mathrm{ch}(\chi_S) = L_{S,n}(x), \tag{3.4}$$

and we extend it to the set of all characters of representations of $\mathcal{H}_n(0)$ by linearity. We let χ_P denote the character of the defining representation of our local $\mathcal{H}_n(0)$ action on the space $\mathbb{C}\mathcal{M}(P)$.

Proposition 3.2.1. *Let P be a finite bounded graded poset of rank n with a snelling. Then the local $\mathcal{H}_n(0)$ action on the maximal chains of P has the property that*

$$\omega F_P(x) = \mathrm{ch}(\chi_P). \tag{3.5}$$

Proof. It is sufficient to show that the coefficient of $L_{S,n}$ for any $S \subseteq [n-1]$ is the same for both sides of (3.5). By (3.3),

$$[L_{S,n}]\,\omega F_P(x) = \beta_P(S^c)$$

where S^c denotes $[n-1] - S$.

Let $J \subseteq [n-1]$ and let $\{i_1, i_2, \ldots, i_k\}$ be a multiset on J where each element of J appears at least once. Let $\mathfrak{m} \in \mathcal{M}(P)$. If $U_i(\mathfrak{m}) \neq \mathfrak{m}$ for some $i \in [n-1]$ then $U_i(\mathfrak{m})$ has one less inversion than $\mathfrak{m}$. It follows that $U_{i_1}U_{i_2}\cdots U_{i_k}(\mathfrak{m}) = \mathfrak{m}$ if and only if the descent set of $\mathfrak{m}$ is disjoint from J. Therefore

$$\begin{aligned} \chi_P(U_{i_1}U_{i_2}\cdots U_{i_k}) &= \#\{\mathfrak{m} \in \mathcal{M}(P) : \mathfrak{m} \text{ has no descents in } J\} \\ &= \sum_{S \supseteq J} \#\{\mathfrak{m} \in \mathcal{M}(P) : \mathfrak{m} \text{ has descent set } S^c\} \\ &= \sum_{S \supseteq J} \beta_P(S^c) \quad \text{by Theorem 2.2.4} \\ &= \sum_{S \subseteq [n-1]} \beta_P(S^c)\chi_S(U_{i_1}U_{i_2}\cdots U_{i_k}). \end{aligned}$$

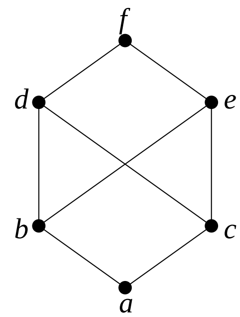


Figure 3-1: A poset with a good $\mathcal{H}_n(0)$ action

$\mathfrak{m}$	$U_1(\mathfrak{m})$	$U_2(\mathfrak{m})$
$\mathfrak{m}_1 : a < b < d < f$	$\mathfrak{m}_3$	$\mathfrak{m}_2$
$\mathfrak{m}_2 : a < b < e < f$	$\mathfrak{m}_4$	$\mathfrak{m}_2$
$\mathfrak{m}_3 : a < c < d < f$	$\mathfrak{m}_3$	$\mathfrak{m}_4$
$\mathfrak{m}_4 : a < c < e < f$	$\mathfrak{m}_4$	$\mathfrak{m}_4$

Table 3.1: A good $\mathcal{H}_n(0)$ action

Thus

$$[L_{S,n}]\,\mathrm{ch}(\chi_P) = [L_{S,n}]\,\mathrm{ch}\left(\sum_{S\subseteq[n-1]} \beta_P(S^c)\chi_S\right) = \beta_P(S^c)$$

as required. □

To summarize, we have that if P is a finite bounded graded poset of rank n with a snelling, then P has a local $\mathcal{H}_n(0)$ action with the property that $\omega F_P(x) = \mathrm{ch}(\chi_P)$. Following [34] and [39], we call such an action a *good* $\mathcal{H}_n(0)$ action. It is natural to ask for a classification of all posets with good $\mathcal{H}_n(0)$ actions. As a starting point, by Theorem 2.4.7, we know that a supersolvable lattice has a good $\mathcal{H}_n(0)$ action. Is there an analogue of lattice supersolvability for posets that need not be lattices that implies the existence of a snelling and hence the existence of a good $\mathcal{H}_n(0)$ action? We will address this question in Chapter 4. Are there bounded graded posets, other than those with snellings, that have good $\mathcal{H}_n(0)$ actions? We answer this question affirmatively in the following example.

Example 3.2.2. Consider the poset P shown in Figure 3-1. As shown in Example 2.3.2, this poset is not snellable. However, it does have a good $\mathcal{H}_n(0)$ action as described in Table 3.1. It is easy to check that this gives a local $\mathcal{H}_3(0)$ action. Now $\alpha_P(\emptyset) = 1, \alpha_P(\{1\}) = 2, \alpha_P(\{2\}) = 2$ and $\alpha_P(\{1,2\}) = 4$. By (2.1), this gives that $\beta_P(S) = 1$ for all $S \subseteq \{1,2\}$. Also, we can check that χ_P decomposes into characters of irreducible representations as

$$\chi_P = \chi_\emptyset + \chi_{\{1\}} + \chi_{\{2\}} + \chi_{\{1,2\}}.$$

By (3.3) and (3.4), this gives

$$\omega F_P(x) = L_{\emptyset,3} + L_{\{1\},3} + L_{\{2\},3} + L_{\{1,2\},3} = \mathrm{ch}(\chi_P).$$

Therefore, this poset has a good $\mathcal{H}_n(0)$ action.

We conclude that it is certainly not true that a bounded graded poset has a good $\mathcal{H}_n(0)$ action if and only if it has a snelling.

Definition 3.2.3. A graded poset P is said to be *bowtie-free* if it does not contain distinct elements a, b, c and d such that a covers both c and d, and such that b covers both c and d.

In particular, all lattices are bowtie-free but the poset shown in Figure 3-1 is not. In Section 3.3, we will prove our second main result:

Theorem 3.2.4. *Let P be a finite bounded poset of rank n that is bowtie-free. Then P has a good $\mathcal{H}_n(0)$ action if and only if P is S_n EL-shellable.*

We get the following immediate corollary:

Corollary 3.2.5. *Let L be a finite graded lattice of rank n. Then the following are equivalent:*

1. *L is supersolvable,*
2. *L is S_n EL-shellable,*
3. *L has a good $\mathcal{H}_n(0)$ action.*

3.3 S_n EL-labellings from good $\mathcal{H}_n(0)$ actions

Our goal for this section is to prove Theorem 3.2.4. Throughout this section, P will denote a finite bounded graded poset of rank n that is bowtie-free. We suppose that P has a good $\mathcal{H}_n(0)$ action and we let χ_P denote the character of the defining representation of this action on the space $\mathbb{C}\mathcal{M}(P)$. In other words, we suppose that there exist functions $U_1, U_2, \ldots, U_{n-1} : \mathcal{M}(P) \to \mathcal{M}(P)$ satisfying the following properties:

1. The action of $U_1, U_2, \ldots, U_{n-1}$ is local.
2. ${U_i}^2 = U_i$ for $i = 1, 2, \ldots, n-1$.
3. $U_iU_j = U_jU_i$ if $|i-j| \geq 2$.
4. $U_iU_{i+1}U_i = U_{i+1}U_iU_{i+1}$ for $i = 1, 2, \ldots, n-2$.
5. $\omega F_P(x) = \mathrm{ch}(\chi_P)$.

As we have previously suggested, given any maximal chain $\mathfrak{m}$ of P, we define the descent set of $\mathfrak{m}$ to be the set

$$\{i \in [n-1] : U_i(\mathfrak{m}) \neq \mathfrak{m}\}.$$

We wish to show that P is snellable. First, we outline our general approach. Suppose P has a unique maximal chain M with empty descent set. Given a maximal chain $\mathfrak{m}$ of P, suppose we can find $U_{i_1}, U_{i_2}, \ldots, U_{i_r}$ with r minimal such that

$$U_{i_1} U_{i_2} \cdots U_{i_r}(\mathfrak{m}) = M. \tag{3.6}$$

Then to $\mathfrak{m}$ we associate the permutation $\gamma_{\mathfrak{m}} = s_{i_1} s_{i_2} \cdots s_{i_r}$ and we label the edges of $\mathfrak{m}$ by $\gamma_{\mathfrak{m}}(1), \gamma_{\mathfrak{m}}(2), \ldots, \gamma_{\mathfrak{m}}(n)$ from bottom to top. Our proof of the validity of this approach divides into four main parts. The first task is to show that M exists and is unique. The next is to show that, given $\mathfrak{m}$, we can always find $U_{i_1}, U_{i_2}, \ldots, U_{i_r}$ satisfying (3.6). The third task is to show that $\gamma_{\mathfrak{m}}$ is well-defined. Finally, we must show that this gives a snelling for P.

Definition 3.3.1. Given maximal chains $\mathfrak{m}$ and $\mathfrak{m}'$ of P, we say that the expression $U_{i_1} U_{i_2} \cdots U_{i_r}(\mathfrak{m}) = \mathfrak{m}'$ is *restless* if $U_{i_r}(\mathfrak{m}) \neq \mathfrak{m}$ and if

$$U_{i_j} U_{i_{j+1}} \cdots U_{i_r}(\mathfrak{m}) \neq U_{i_{j+1}} \cdots U_{i_r}(\mathfrak{m}) \quad \text{for } j = 1, 2, \ldots, r-1.$$

We say that two sequences $U_{i_1} U_{i_2} \cdots U_{i_r}$ and $U_{j_1} U_{j_2} \cdots U_{j_r}$ are in the same *braid class* if we can get from one to the other by applying Properties 3 and 4 repeatedly. It is not difficult to see that if $U_{i_1} U_{i_2} \cdots U_{i_r}$ and $U_{j_1} U_{j_2} \cdots U_{j_r}$ are in the same braid class and if $U_{i_1} U_{i_2} \cdots U_{i_r}(\mathfrak{m}) = \mathfrak{m}'$ is restless, then $U_{j_1} U_{j_2} \cdots U_{j_r}(\mathfrak{m}) = \mathfrak{m}'$ is restless. Indeed, it is easy to see that applying Property 3 to a restless sequence will give another restless sequence. Using the fact that P is bowtie-free, it can be readily checked that applying Property 4 to a restless sequence also gives another restless sequence.

To every sequence $i_1, i_2, \ldots, i_r$ such that $U_{i_1} U_{i_2} \cdots U_{i_r}(\mathfrak{m}) = \mathfrak{m}'$, we can associate a counting vector of length $n-1$ where the jth coordinate equals the number of times that i_j appears in the sequence $i_1, i_2, \ldots, i_r$. We say that the expression $U_{i_1} U_{i_2} \cdots U_{i_r}(\mathfrak{m}) = \mathfrak{m}'$ is *lexicographically minimal* (or *lex. minimal* for short) if no sequence $U_{j_1} U_{j_2} \cdots U_{j_r}$ in the braid class of $U_{i_1} U_{i_2} \cdots U_{i_r}$ and satisfying $U_{j_1} U_{j_2} \cdots U_{j_r}(\mathfrak{m}) = \mathfrak{m}'$ has a lexicographically less counting vector.

The following result will help us to complete our first two tasks.

Lemma 3.3.2. *Let $\mathfrak{m}'$ be any maximal chain of P. Suppose $U_i(\mathfrak{m}') \neq \mathfrak{m}'$. Then there do not exist $i_1, i_2, \ldots, i_r$ satisfying $U_{i_1} U_{i_2} \cdots U_{i_r} U_i(\mathfrak{m}') = \mathfrak{m}'$.*

Proof. Suppose there exists a sequence $i_1, i_2, \ldots, i_r$ satisfying $U_{i_1} U_{i_2} \cdots U_{i_r} U_i(\mathfrak{m}') = \mathfrak{m}'$. It suffices to consider the case when $U_{i_1} U_{i_2} \cdots U_{i_r} U_i(\mathfrak{m}') = \mathfrak{m}'$ is restless and lex. minimal. Let $l \in [n-1]$ denote the minimum element of the sequence $i_1, i_2, \ldots, i_r, i$. Since our equation is restless U_l must occur at least twice in the sequence.

Take any pair of U_l appearances with no U_l between them. If we had no U_{l+1} between them, we could apply Property 3 until we had an appearance of U_lU_l, contradicting the restless property since $U_l^2 = U_l$. If there is just one U_{l+1} between them, we can apply Property 3 to get $U_lU_{l+1}U_l$ appearing and then apply Property 4 to get $U_{l+1}U_lU_{l+1}$, contradicting the lex. minimal property. We conclude that, between the two appearances of U_l, there are at least two appearances of U_{l+1}. Choose any two of these appearances of U_{l+1} that don't have another U_{l+1} between them and apply the same argument to show that there must be at least two appearances of U_{l+2} between them. Repeating this process, we eventually get U_iU_i appearing, yielding a contradiction. □

More generally, we can apply the same argument to prove the following statement:

Lemma 3.3.3. *Suppose $U_{i_1}U_{i_2}\cdots U_{i_r}(\mathfrak{m}) = M$ is restless and lex. minimal. Let l denote the minimum element of the sequence $i_1, i_2, \ldots, i_r$. Then U_l appears exactly once and for $l < i \leq n-1$, there must be an appearance of U_{i-1} between any two appearances of U_i.*

The following result is essentially a rephrasing of Property 5 of our good $\mathcal{H}_n(0)$ action into more amenable terms.

Proposition 3.3.4. *For all $S \subseteq [n-1]$, $\alpha_P(S)$ equals the number of maximal chains of P with descent set contained in S.*

Proof. We know that

$$\chi_P = \sum_{S\subseteq[n-1]} b_{P,S}\chi_S$$

for some set of coefficients $\{b_{P,S}\}_{S\subseteq[n-1]}$ and hence

$$\mathrm{ch}(\chi_P) = \sum_{S\subseteq[n-1]} b_{P,S}L_{S,n}.$$

By (3.3) and Property 5, we see that $b_{P,S} = \beta_P(S^c)$.

Now let $T = \{i_1, i_2, \ldots, i_k\} \subseteq [n-1]$. Then

$$\begin{aligned}\sum_{S\supseteq T}\beta_P(S^c) &= \sum_{S\subseteq[n-1]}\beta_P(S^c)\chi_S(U_{i_1}U_{i_2}\cdots U_{i_k})\\ &= \chi_P(U_{i_1}U_{i_2}\cdots U_{i_k})\\ &= \#\{\mathfrak{m}\in\mathcal{M}(P) : \mathfrak{m} \text{ has no descents in } T\}\end{aligned}$$

by Lemma 3.3.2. Therefore,

$$\sum_{S\supseteq T}\beta_P(S^c) = \sum_{S\supseteq T}\#\{\mathfrak{m}\in\mathcal{M}(P) : \mathfrak{m} \text{ has descent set } S^c\}$$

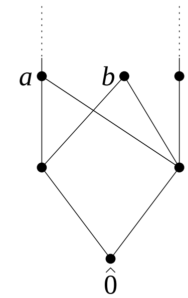


Figure 3-2: A portion of P

which is equivalent to

$$\sum_{S^c \subseteq [n-1]-T} \beta_P(S^c) = \sum_{S^c \subseteq [n-1]-T} \#\{\mathfrak{m} \in \mathcal{M}(P) : \mathfrak{m} \text{ has descent set } S^c\}.$$

By (2.2), this gives that

$$\alpha_P([n-1]-T) = \#\{\mathfrak{m} \in \mathcal{M}(P) : \mathfrak{m} \text{ has descent set contained in } [n-1]-T\}$$

for all $T \subseteq [n-1]$. Setting $S = [n-1]-T$, we get the required result. □

In particular, setting $S = \emptyset$, we see that P has exactly one maximal chain, which we denote by M, with no descents. Also, given a maximal chain $\mathfrak{m}$ of P, by Lemma 3.3.2 and the finiteness of P, we can find $U_{i_1}, U_{i_2}, \ldots, U_{i_r}$ with r minimal such that $U_{i_1}U_{i_2}\cdots U_{i_r}(\mathfrak{m}) = M$. This completes our first two tasks.

Given any maximal chain $\mathfrak{m}$ of P, we consider the braid classes of the set of sequences $U_{i_1}U_{i_2}\cdots U_{i_r}$ such that $U_{i_1}U_{i_2}\cdots U_{i_r}(\mathfrak{m}) = M$ is restless. Our next task is to show that there is only one such braid class. Every braid class contains at least one element $U_{i_1}U_{i_2}\cdots U_{i_r}$ such that $U_{i_1}U_{i_2}\cdots U_{i_r}(\mathfrak{m}) = M$ is restless and lex. minimal. For such an element, the minimum, l, of $i_1, i_2, \ldots, i_r$ is the lowest rank for which $\mathfrak{m} \neq \mathfrak{m}_0$, by Lemma 3.3.3. It follows that l is the same for all the braid classes. It suffices to consider the case when $l = 1$.

The following result is central to our proof that there is just one braid class.

Lemma 3.3.5. *Suppose that the expressions $U_{i_1}U_{i_2}\cdots U_{i_r}(\mathfrak{m}) = M$ and $U_{j_1}U_{j_2}\cdots U_{j_s}(\mathfrak{m}) = M$ are both restless. Then there exists an element of the braid class of $U_{i_1}U_{i_2}\cdots U_{i_r}$ and an element of the braid class of $U_{j_1}U_{j_2}\cdots U_{j_s}$ both ending on the right with the same U_i.*

Proof. Suppose $U_{i_1}U_{i_2}\cdots U_{i_r}$ and $U_{j_1}U_{j_2}\cdots U_{j_s}$ are in different braid classes. Without loss of generality, we take them both to be lex. minimal. If U_1 can be moved to the right-hand end in both by applying Property 3, then there's nothing to prove. Suppose, by applying Property 3, that U_1 can be brought to the right end in one sequence but not in the other. Then P must have the edges shown in Figure 3-2, where $\mathfrak{m}$ and M are the maximal chains on the left and right, respectively. We see that we get a contradiction with the bowtie-free property unless $a = b$. In this case,

U_2 appears at least twice in the latter sequence to the right of the unique appearance of U_1, contradicting Lemma 3.3.3. We conclude that U_1 can't be brought to the right end in either sequence. Now we consider that portion of each sequence to the right of the unique U_1. By the same logic, the maximal chains we get when we apply these portions to $\mathfrak{m}$ must have the same element at rank 2.

Consider the unique U_2 in each of these portions. By a similar argument, we conclude that either we've nothing to prove or else U_2 can't be brought to the right of either sequence by applying Property 3. In the latter case, we consider the portion of each sequence to the right of the unique U_2. The maximal chains we get when we apply these portions to $\mathfrak{m}$ must have the same element at rank 3. Repeating the same argument, we are eventually reduced to the case where U_i is the element at the right end of both sequences, for some i. □

Proposition 3.3.6. *If the expressions $U_{i_1}U_{i_2}\cdots U_{i_r}(\mathfrak{m}) = M$ and $U_{j_1}U_{j_2}\cdots U_{j_s}(\mathfrak{m}) = M$ are both restless then $s_{i_1}s_{i_2}\cdots s_{i_r} = s_{j_1}s_{j_2}\cdots s_{j_s}$.*

Proof. It suffices to prove the result in the case when r is as small as possible. We prove the result by induction on r, the result being trivially true when $r = 0$.

For $r > 0$, by the previous lemma, there exists an element $U_{I_1}U_{I_2}\cdots U_{I_{r-1}}U_i$ of the braid class of $U_{i_1}U_{i_2}\cdots U_{i_r}$ and an element $U_{J_1}U_{J_2}\cdots U_{J_{s-1}}U_i$ of the braid class of $U_{j_1}U_{j_2}\cdots U_{j_s}$. Consider $U_i(\mathfrak{m})$. By the induction hypothesis,

$$s_{I_1}s_{I_2}\cdots s_{I_{r-1}} = s_{J_1}s_{J_2}\cdots s_{J_{s-1}}.$$

Therefore, since permutations are invariant under braid moves,

$$s_{i_1}s_{i_2}\cdots s_{i_r} = s_{I_1}s_{I_2}\cdots s_{I_{r-1}}s_i = s_{J_1}s_{J_2}\cdots s_{J_{s-1}}s_i = s_{j_1}s_{j_2}\cdots s_{j_s}.$$

□

Finally, we can make the following definition:

Definition 3.3.7. If $U_{i_1}U_{i_2}\cdots U_{i_r}(\mathfrak{m}) = M$ is restless then we define

$$\gamma_{\mathfrak{m}} = s_{i_1}s_{i_2}\cdots s_{i_r}.$$

For every maximal chain $\mathfrak{m}$ of P, we label the edges of $\mathfrak{m}$ from bottom to top by $\gamma_{\mathfrak{m}}(1), \gamma_{\mathfrak{m}}(2), \ldots, \gamma_{\mathfrak{m}}(n)$. Our final task is to show that this gives an edge labelling, and in particular a snelling, for P. We divide the proof into a number of small steps.

Step 1. *If $U_{i_1}U_{i_2}\cdots U_{i_r}(\mathfrak{m}) = M$ is restless then $\gamma_{\mathfrak{m}} = s_{i_1}s_{i_2}\cdots s_{i_r}$ is a reduced expression. Furthermore, if $\gamma_{\mathfrak{m}} = s_{j_1}s_{j_2}\cdots s_{j_r}$ is another reduced expression, then $U_{j_1}U_{j_2}\cdots U_{j_r}(\mathfrak{m}) = M$ is restless.*

Proof. The first assertion follows from the fact that if $\gamma_{\mathfrak{m}} = s_{i_1}s_{i_2}\cdots s_{i_r}$ is not reduced then we can apply a sequence of braid moves to get s_is_i appearing. This contradicts the restless property. The second assertion follows from Tits' Word Theorem applied to the reduced expressions for $\gamma_{\mathfrak{m}}$, together with the fact that applying Properties 3 and 4 to a restless expression results in another restless expression. □

Step 2. *The permutation* $\gamma_{\mathfrak{m}}$ *has a descent at* i *if and only if* $U_i(\mathfrak{m}) \neq \mathfrak{m}$. *In this case,* $\gamma_{U_i(\mathfrak{m})}$ *is the same as* $\gamma_{\mathfrak{m}}$ *except that the* i*th and* $(i+1)$*st elements have been switched, removing the descent.*

Proof. Using Step 1, we have that

$$\begin{aligned} & U_i(\mathfrak{m}) \neq \mathfrak{m} \\ \Leftrightarrow \quad & U_{i_1}U_{i_2}\cdots U_{i_r}U_i(\mathfrak{m}) = M \text{ is restless for some } i_1, i_2, \ldots, i_r \\ \Leftrightarrow \quad & s_{i_1}s_{i_2}\cdots s_{i_r}s_i = \gamma_{\mathfrak{m}} \text{ is a reduced expression for some } i_1, i_2, \ldots, i_r \\ \Leftrightarrow \quad & \gamma_{\mathfrak{m}}s_i \text{ has one less inversion than } \gamma_{\mathfrak{m}} \\ \Leftrightarrow \quad & \gamma_{\mathfrak{m}} \text{ has a descent at } i. \end{aligned}$$

When $U_i(\mathfrak{m}) \neq \mathfrak{m}$ and $\gamma_{\mathfrak{m}} = s_{i_1}s_{i_2}\cdots s_{i_r}s_i$ is reduced we see that $\gamma_{U_i(\mathfrak{m})} = s_{i_1}s_{i_2}\cdots s_{i_r}$, yielding the second statement. □

Step 3. *Let* $S \subseteq [n-1]$. *Then every chain in* P *with rank set equal to* S *has exactly one extension to a maximal chain of* P *with descent set contained in* S.

Proof. Given any chain $\mathfrak{c}$ with rank set S, let $\mathfrak{m}$ be any extension of $\mathfrak{c}$ to a maximal chain in P. Apply U_i for $i \notin S$ repeatedly to $\mathfrak{m}$. By Step 2, this will eventually yield an extension of $\mathfrak{c}$ which is a maximal chain with descent set contained in S. Therefore, every chain with rank set S has at least one such extension. We get

$$\alpha_P(S) \leq \#\left\{\mathfrak{m} \in \mathcal{M}(P) : \mathfrak{m} \text{ has descent set contained in } S\right\}.$$

However, by Proposition 3.3.4,

$$\alpha_P(S) = \#\left\{\mathfrak{m} \in \mathcal{M}(P) : \mathfrak{m} \text{ has descent set contained in } S\right\}.$$

Thus $\mathfrak{c}$ has exactly one extension to a maximal chain of P with descent set contained in S. □

Step 4. *For every maximal chain* $\mathfrak{m}$ *of* P, *labelling the edges of* $\mathfrak{m}$ *from bottom to top by* $\gamma_{\mathfrak{m}}(1), \gamma_{\mathfrak{m}}(2), \ldots, \gamma_{\mathfrak{m}}(n)$ *gives an edge labelling for* P.

Proof. Let $x, y \in P$ be such that y covers x and let $\mathfrak{m}$ and $\mathfrak{m}'$ be maximal chains of P containing both x and y. Define $S = [\text{rk}(x), \text{rk}(y)]$ and let $\mathfrak{m}_{(x,y)}$ denote the unique extension of $x < y$ to a maximal chain with descent set contained in S. By applying U_i for $i \notin S$ repeatedly to $\mathfrak{m}$, we can reach $\mathfrak{m}_{(x,y)}$. By Step 2, since we never apply $U_{\text{rk}(x)}$ or $U_{\text{rk}(y)}$ in reaching $\mathfrak{m}_{(x,y)}$ from $\mathfrak{m}$, we must have $\gamma_{\mathfrak{m}}(\text{rk}(y)) = \gamma_{\mathfrak{m}_{(x,y)}}(\text{rk}(y))$. That is, $\mathfrak{m}$ and $\mathfrak{m}_{(x,y)}$ give the same label to the edge (x, y). Similarly, $\mathfrak{m}'$ and $\mathfrak{m}_{(x,y)}$ give the same label to the edge (x, y). Therefore, $\mathfrak{m}$ and $\mathfrak{m}'$ give the same label to the edge (x, y) and so we have an edge labelling for P. □

Step 5. *This edge labelling is a snelling for* P.

Proof. Let $x, y \in P$ be such that $x < y$. Let

$$S = [n-1] - \{\text{rk}(x)+1, \text{rk}(x)+2, \ldots, \text{rk}(y)-1\}$$

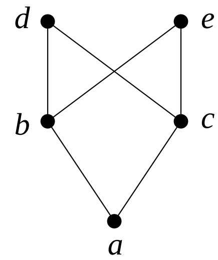


Figure 3-3: A “mask”

in Step 3. The fact that the interval $[x, y]$ has exactly one increasing maximal chain follows from Step 3 and the fact that we now have an edge labelling. Every maximal chain is labelled by a permutation by definition. Therefore, P is snellable, proving Theorem 3.2.4. □

3.4 Remarks and examples

Remark 3.4.1. Theorem 3.2.4 does indeed contain information not contained in Corollary 3.2.5, in that there exist bounded graded bowtie-free posets that are snellable but are not lattices. For example, take the lattice B_4 with a snelling as described in Example 2.2.2. Now delete the edge $(\{3,4\},\{2,3,4\})$ in the Hasse diagram of B_4 to form the Hasse diagram of a new poset P and label the remaining edges as they are labelled in B_4. We can check that the new poset has the desired properties. Indeed, since B_4 is bowtie-free, P must be bowtie-free. The vertices corresponding to $\{3\}$ and $\{4\}$ have $\{3,4\}$, $\{1,3,4\}$, $\{2,3,4\}$ and $\{1,2,3,4\}$ as common upper bounds but now have no least upper bound since $\{3,4\}$ is no longer less that $\{2,3,4\}$. Finally, there are no increasing chains in B_4 of length greater than 1 that contain both $\{3,4\}$ and $\{2,3,4\}$ and so every interval of P still has exactly one increasing chain.

Remark 3.4.2. A careful examination of the proof of Theorem 3.2.4 reveals that, rather than working with a bowtie-free poset, the same proof will also work for a poset that is free of the shape shown in Figure 3-3. We will refer to such a poset as a “mask-free” poset. (Admittedly, the origin of this term requires some imagination!) More precisely, we will say that a poset is mask-free if it does not contain elements a, b, c, d and e with relations $a \lessdot b$ (and not just $a < b$), $a \lessdot c$, $b \lessdot d$, $b \lessdot e$, $c \lessdot d$ and $c \lessdot e$. Using a dual argument in the proof also allows it to work for posets that are free of the dual of the shape shown in Figure 3-3. We will refer to such posets as “mask*-free” posets. Therefore, we have actually proved the following result: a finite bounded poset of rank n that is mask-free or mask*-free has a good $\mathcal{H}_n(0)$ action if and only if it has an S_n EL-labelling. However, at this point, we cannot say anything interesting or relevant about posets that are mask-free but not bowtie-free. Therefore, for the purposes of clarity, we stated Theorem 3.2.4 only for bowtie-free posets.

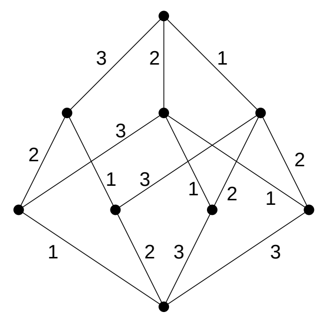


Figure 3-4: A non-lattice with a snelling

Remark 3.4.3. It seems that we have fully answered the question of bounded graded posets in the bowtie-free case. What can we say about such posets that are not bowtie-free? In Example 3.2.2 we saw a poset with a bowtie that has a good $\mathcal{H}_n(0)$ action but which is not snellable. On the other hand, Figure 3-4 shows a bounded graded poset that has a bowtie but which is still snellable and hence, by Proposition 3.2.1, has a good $\mathcal{H}_n(0)$ action. This suggests the following question.

Question 3.4.4. Let $\mathcal{C}$ denote the class of bounded graded posets that have a good $\mathcal{H}_n(0)$ action. Is there some "nice" characterization of $\mathcal{C}$, possibly in terms of edge labellings?

Chapter 4

Connections with left modularity and generalizations

4.1 Left modularity

Corollary 3.2.5 gives two new characterizations of lattice supersolvability. We now introduce a third new characterization.

Given an element x of a finite lattice L, and a pair of elements $y \leq z$, it is always true that

$$(x \vee y) \wedge z \geq (x \wedge z) \vee y. \tag{4.1}$$

Definition 4.1.1. An element x of a lattice L is said to be *left modular* if, for all $y \leq z$ in L, we have

$$(x \vee y) \wedge z = (x \wedge z) \vee y. \tag{4.2}$$

We will say that a chain in L is *left modular* if each of its elements is left modular. In general, if a property is defined for elements of a poset, then when we say that a chain has that property, we mean that every element of the chain has that property.

Example 4.1.2. If L is a distributive lattice, then

$$(x \wedge z) \vee y = (x \vee y) \wedge (z \vee y)$$

for all elements x, y and z of L. If we assume that $y \leq z$, then $z \vee y = z$ and we get Equation (4.2). Therefore, every element of a distributive lattice is left modular.

The following result appears as [36, Proposition 2.2].

Proposition 4.1.3. *Any M-chain of a lattice L is a left modular maximal chain.*

Proof. Let x be any element of an M-chain M of a lattice L and let $y \leq z$ in L. The sublattice of L generated by M, y and z is distributive. Thus, as shown in Example 4.1.2, x is left modular. We conclude that M is left modular. □

The following observations leading to the theorem below were made by Hugh Thomas. It is shown by Larry Shu-Chung Liu in [21, §3.2] that if a lattice has a left modular maximal chain of length n, then it has an EL-labelling where all the labels are from the set $[n]$ and no label appears twice on any maximal chain. Furthermore, the labels on the left modular maximal chain are increasing. Restricting to graded lattices, this tells us that if a lattice has a left modular maximal chain, then it has an S_n EL-labelling. Combining this with Corollary 3.2.5 and Proposition 4.1.3, we get the following theorem.

Theorem 4.1.4. *Let L be a finite graded lattice of rank n. Then the following are equivalent:*

1. *L is supersolvable,*
2. *L is S_n EL-shellable,*
3. *L has a good $\mathcal{H}_n(0)$ action,*
4. *L has a left modular maximal chain.*

It is shown in [36] that if L is upper-semimodular, then possessing a left modular maximal chain and being supersolvable are equivalent. Theorem 4.1.4 is a considerable strengthening of this. Here we used S_n EL-labellings to connect left modularity and supersolvability. It is natural to ask for a direct proof of the equivalence of (1) and (4).

Note. For each of these four properties, there is a unique distinguished maximal chain. Respectively, the distinguished chain is the M-chain, the increasing chain in the snelling, the chain fixed under the good $\mathcal{H}_n(0)$ action and the left modular maximal chain. We should highlight the fact that if a maximal chain is distinguished with respect to one property, then it is a distinguished chain for the other three properties. More precisely, the proofs of the equivalences in Theorem 4.1.4 actually give us the following slightly stronger result:

Porism 4.1.5. *Let L be a finite graded lattice of rank n with a maximal chain M. Then the following are equivalent:*

1. *M is an M-chain,*
2. *M is the increasing maximal chain of an S_n EL-labelling of L,*
3. *M is the unique chain that is fixed under a good $\mathcal{H}_n(0)$ action of L,*
4. *M is left modular.*

A definite strength of Theorem 4.1.4 is that it brings together four different areas of the theory of partially ordered sets. On the other hand, the theorem only applies to graded lattices. The remainder of this chapter will be devoted to extensions of the equivalences to more general classes of posets.

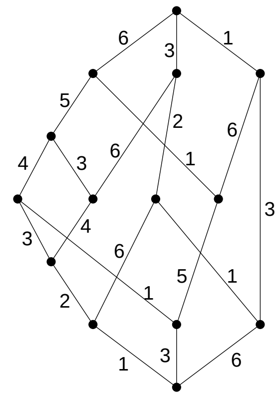


Figure 4-1: The Tamari lattice T_4 and its interpolating EL-labelling

4.2 Interpolating labellings and viability

Observe that the definition of left modularity applies equally well to lattices that are not graded. This suggests that we might try to generalize the definitions of supersolvability, good $\mathcal{H}_n(0)$ actions or S_n EL-labellings to lattices that need not be graded. We will discuss generalizations of supersolvability in Section 4.5. The following generalization of S_n EL-labellings was suggested by Hugh Thomas.

Definition 4.2.1. Let P be a finite bounded poset. An EL-labelling γ of P is said to be *interpolating* if, for any $y \lessdot u \lessdot z$, either

(i) $\gamma(y,u) < \gamma(u,z)$ or

(ii) the increasing chain from y to z, say $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$, has the properties that its labels are strictly increasing and that $\gamma(w_0, w_1) = \gamma(u,z)$ and $\gamma(w_{r-1}, w_r) = \gamma(y,u)$.

Example 4.2.2. The reader is invited to check that the labelling of the non-graded poset shown in Figure 4-1 is an interpolating EL-labelling. In fact, the poset shown is the so-called "Tamari lattice" T_4. For all positive integers n, there exists a Tamari lattice T_n with C_n elements, where $C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. More information on the Tamari lattice can be found in [9, §9], [10, §7] and the references given there, and in [21, §3.2], where this interpolating EL-labelling appears. The Tamari lattice is shown to be EL-shellable in [9] and is shown to have a left modular maximal chain in [10].

To show that interpolating EL-labellings are a suitable generalization of S_n EL-labellings, there are two basic requirements. The first is that an interpolating EL-labelling of a graded poset of rank n should be an S_n EL-labelling. While this is not obvious from the definition, it will follow as a consequence of Lemma 4.4.2, one of our first results on interpolating EL-labellings. The second requirement is that a lattice should have an interpolating EL-labelling if and only if it has a left modular maximal chain. The following result was proved by Thomas.

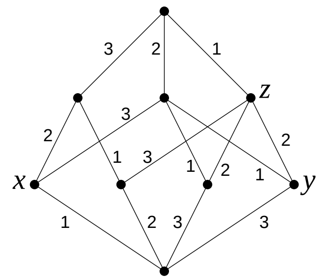


Figure 4-2: A poset with a viable left modular maximal chain

Theorem 4.2.3. *A finite lattice has an interpolating EL-labelling with increasing chain M if and only if M is a left modular maximal chain.*

Rather than proving this result, we would like to formulate an extension of it. The extended version of Theorem 4.2.3 and the other material of this chapter became the subject of a joint paper [28] with Thomas. Observe that we defined interpolating EL-labellings for posets that need not be lattices. This suggests that we might try to generalize the definition of left modularity to posets where meet and join are not always well-defined.

Let P be any bounded poset. Let x and y be elements of P. We know that x and y have at least one common upper bound, namely $\hat{1}$. The set of common upper bounds of x and y might not have a least element but, if it does, then we denote this least element by $x \vee y$. Similarly, if x and y have a greatest common lower bound, then we denote it by $x \wedge y$.

Now let w and z be elements of P with $w, z \geq y$. Consider the set of common lower bounds for w and z that are also greater than or equal to y. Clearly, y is in this set. If this set has a greatest element, then we denote it by $w \wedge_y z$ and we say that $w \wedge_y z$ is well-defined (in $[y, \hat{1}]$). For example, consider the poset shown in Figure 4-2, which we encountered in Remark 3.4.3 as a non-lattice with a snelling. We see that $(x \vee y) \wedge_y z$ is well-defined in this poset, even though $(x \vee y) \wedge z$ is not. Similarly, let w and y be elements of P with $w, y \leq z$. If the set $\{u \in P \mid u \geq w, y \text{ and } u \leq z\}$ has a least element, then we denote it by $w \vee^z y$ and we say that $w \vee^z y$ is well-defined (in $[\hat{0}, z]$). We will usually be interested in expressions of the form $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$. The reader that is solely interested in the lattice case can choose to ignore the subscripts and superscripts on the meet and join symbols.

Definition 4.2.4. An element x of a finite bounded poset P is said to be *viable* if, for all $y \leq z$ in P, $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ are well-defined. A viable element x of P is said to be *left modular* if, for all $y \leq z$ in P,

$$(x \vee y) \wedge_y z = (x \wedge z) \vee^z y.$$

Example 4.2.5. The poset shown in Figure 4-2 is certainly not a lattice but the reader can check that the increasing maximal chain is viable and left modular.

We are now ready to state our extensions of Theorem 4.2.3.

Theorem 4.2.6. *Let P be a bounded poset with a viable left modular maximal chain M. Then P has an interpolating EL-labelling with M as its increasing maximal chain.*

The proof of this theorem will be the content of the next section. In Section 4.4, we will prove the following converse result.

Theorem 4.2.7. *Let P be a bounded poset with an interpolating EL-labelling. The unique increasing chain from $\hat{0}$ to $\hat{1}$ is a viable left modular maximal chain.*

As one consequence, we get Theorem 4.2.3 when P is a lattice. As another consequence, in the graded poset case, we have given an answer to the question of when P has an S_n EL-labelling. As we know from Proposition 3.2.1 and Theorem 3.2.4, this has ramifications on the existence of a good $\mathcal{H}_n(0)$ action.

These two theorems, when compared with Theorem 4.1.4, might lead one to ask about possible supersolvability results for bounded posets that aren't graded lattices. This problem is discussed in Section 4.5. We obtain a satisfactory result in the graded case but the ungraded case is left as an open problem.

4.3 Interpolating labellings from left modularity

Our aim for this section is to prove Theorem 4.2.6. We suppose that P is a bounded poset with a viable left modular maximal chain $M : \hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$. We want to show that P has an interpolating EL-labelling. We begin with some lemmas which build on the viability and left modularity properties.

Lemma 4.3.1. *Suppose that $y \leq w \leq z$ in P and let $x \in M$. Then $((x \wedge z) \vee^z y) \vee^z w$ is well-defined and equals $(x \wedge z) \vee^z w$. Similarly, $((x \vee y) \wedge_y z) \wedge_y w$ is well-defined and equals $(x \vee y) \wedge_y w$.*

Proof. We need to check that, in $[\hat{0}, z]$, $(x \wedge z) \vee^z w$ is the least common upper bound for w and $(x \wedge z) \vee^z y$. Clearly $(x \wedge z) \vee^z w$ is a common upper bound for w and $(x \wedge z) \vee^z y$. Now suppose $u \in [\hat{0}, z]$ is a common upper bound for w and $(x \wedge z) \vee^z y$. In particular, $u \geq x \wedge z$ and $u \geq w$. Therefore, $u \geq (x \wedge z) \vee^z w$ and so $(x \wedge z) \vee^z w$ is the least common upper bound.

Similarly, in $[y, \hat{1}]$, $(x \vee y) \wedge_y w$ is the greatest common lower bound for $(x \vee y) \wedge_y z$ and w. □

Lemma 4.3.2. *Suppose that $t \leq u$ in $[y, z]$ and $x \in M$. Let $w = (x \vee y) \wedge_y z = (x \wedge z) \vee^z y$ in $[y, z]$. Then $(w \vee^z t) \wedge_t u$ and $(w \wedge_y u) \vee^u t$ are well-defined elements of $[t, u]$ and are equal.*

Proof. We see that, by Lemma 4.3.1,

$$\begin{aligned}(x \vee t) \wedge_t u &= ((x \vee t) \wedge_t z) \wedge_t u = ((x \wedge z) \vee^z t) \wedge_t u \\ &= (((x \wedge z) \vee^z y) \vee^z t) \wedge_t u = (w \vee^z t) \wedge_t u.\end{aligned}$$

Similarly,

$$(x \wedge u) \vee^u t = (w \wedge_y u) \vee^u t$$

But $(x \vee t) \wedge_t u = (x \wedge u) \vee^u t$, yielding the result. □

Lemma 4.3.3. *Suppose x and w are viable and that x is left modular in P.*

(a) If $x \lessdot w$ then for any z in P we have $x \wedge z \leq w \wedge z$.

(b) If $w \lessdot x$ then for any y in P we have $w \vee y \leq x \vee y$.

Part (b) appears in the lattice case as [21, Lemma 2.5.6] and as [22, Lemma 5.3].

Proof. We prove (a); (b) is similar. Assume, seeking a contradiction, that $x \wedge z < u < w \wedge z$ for some $u \in P$. Now $u \leq z$ and $u \leq w$. It follows that $u \not\leq x$.

Now $x < x \vee u \leq w$. Therefore, $w = x \vee u$. So

$$u = (x \wedge z) \vee^z u = (x \vee u) \wedge_u z = w \wedge z,$$

which is a contradiction. □

We now prove a slight extension of [21, Lemma 2.5.7] and [22, Lemma 5.4].

Lemma 4.3.4. *The elements of $[y, z]$ of the form $(x_i \vee y) \wedge_y z$ form a viable left modular maximal chain in $[y, z]$.*

Proof. Lemma 4.3.2 gives the viability and left modularity properties. By Lemma 4.3.3(b), $x_i \vee y \leq x_{i+1} \vee y$. By Lemma 4.3.2 with $z = \hat{1}$, we have that $x_i \vee y$ is left modular in $[y, \hat{1}]$. Therefore, $(x_i \vee y) \wedge_y z \leq (x_{i+1} \vee y) \wedge_y z$ by Lemma 4.3.3(a). □

We are now ready to specify an edge labelling for P. Let P be a bounded poset with a viable left modular maximal chain $M : \hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$. We choose a label set $l_1 < \cdots < l_n$ of natural numbers. (For most purposes, we can let $l_i = i$.) We define an edge labelling γ on P by, for $y \lessdot z$, $\gamma(y, z) = l_i$ if

$$(x_{i-1} \vee y) \wedge_y z = y \quad \text{and} \quad (x_i \vee y) \wedge_y z = z.$$

It is easy to see that γ is well-defined, and this also follows from the next lemma. We will refer to it as the labelling *induced* by M and the label set $\{l_i\}$. When P is a lattice, this labelling appears, for example, in [21] and [44]. As in [21], we can give an equivalent definition of γ as follows.

Lemma 4.3.5. *Suppose $y \lessdot z$ in P. Then $\gamma(y, z) = l_i$ if and only if*

$$i = \min\{j \mid x_j \vee y \geq z\} = \max\{j + 1 \mid x_j \wedge z \leq y\}.$$

Proof. That $i = \min\{j \mid x_j \vee y \geq z\}$ is immediate from the definition of γ. By left modularity, $\gamma(y, z) = l_i$ if and only if $(x_{i-1} \wedge z) \vee^z y = y$ and $(x_i \wedge z) \vee^z y = z$. In other words, $x_{i-1} \wedge z \leq y$ and $x_i \wedge z \not\leq y$. It follows that $i = \max\{j + 1 \mid x_j \wedge z \leq y\}$. □

We are now ready for the last, and most important, of our preliminary results. Let $[y,z]$ be an interval in P. We call the maximal chain of $[y,z]$ from Lemma 4.3.4 the *induced left modular maximal chain* of $[y,z]$. One way to get a second edge labelling for $[y,z]$ would be to take the labelling induced in $[y,z]$ by this induced maximal chain. We now prove that, for a suitable choice of label set, this labelling coincides with γ.

Proposition 4.3.6. *Let P be a bounded poset, $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ a viable left modular maximal chain and γ the corresponding edge labelling with label set $\{l_i\}$. Let $y < z$, and define c_i by saying*

$$\begin{aligned} y &= (x_0 \vee y) \wedge_y z = \cdots = (x_{c_1 - 1} \vee y) \wedge_y z \\ &\lessdot (x_{c_1} \vee y) \wedge_y z = \cdots = (x_{c_2 - 1} \vee y) \wedge_y z \lessdot \cdots \\ &\lessdot (x_{c_r} \vee y) \wedge_y z = \cdots = (x_n \vee y) \wedge_y z. \end{aligned}$$

Let $m_i = l_{c_i}$. Let δ be the labelling of $[y,z]$ induced by its induced left modular maximal chain and the label set $\{m_i\}$. Then δ agrees with γ restricted to the edges of $[y,z]$.

Proof. Suppose $t \lessdot u$ in $[y,z]$. Then we have

$$\begin{aligned} \delta(t,u) = m_i \quad &\Leftrightarrow \quad (((x_{c_i - 1} \vee y) \wedge_y z) \vee^z t) \wedge_t u = t \text{ and} \\ &\qquad (((x_{c_i} \vee y) \wedge_y z) \vee^z t) \wedge_t u = u \\ &\Leftrightarrow \quad (((x_{c_i - 1} \wedge z) \vee^z y) \vee^z t) \wedge_t u = t \text{ and} \\ &\qquad (((x_{c_i} \wedge z) \vee^z y) \vee^z t) \wedge_t u = u \\ &\Leftrightarrow \quad ((x_{c_i - 1} \wedge z) \vee^z t) \wedge_t u = t \text{ and } ((x_{c_i} \wedge z) \vee^z t) \wedge_t u = u \\ &\Leftrightarrow \quad ((x_{c_i - 1} \vee t) \wedge_t z) \wedge_t u = t \text{ and } ((x_{c_i} \vee t) \wedge_t z) \wedge_t u = u \\ &\Leftrightarrow \quad (x_{c_i - 1} \vee t) \wedge_t u = t \text{ and } (x_{c_i} \vee t) \wedge_t u = u \\ &\Leftrightarrow \quad \gamma(t,u) = l_{c_i}. \end{aligned}$$

□

Proof of Theorem 4.2.6. We now know that the induced left modular chain in $[y,z]$ has (strictly) increasing labels, say $m_1 < m_2 < \cdots < m_r$. Our first step is to show that it is the only maximal chain with (weakly) increasing labels. Suppose that $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$ is the induced chain and that $y = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_s = z$ is another chain with increasing labels.

If $s = 1$ then $y \lessdot z$ and the result is clear. Suppose $s \geq 2$. By Proposition 4.3.6, we may assume that the labelling on $[y,z]$ is induced by the induced left modular chain $\{w_i\}$. In particular, we have that $\gamma(u_i, u_{i+1}) = m_l$ where $l = \min\{j \mid w_j \vee^z u_i \geq u_{i+1}\}$. Let k be the least number such that $u_k \geq w_1$. Then it is clear that $\gamma(u_{k-1}, u_k) = m_1$. Note that this is the smallest label that can occur on any edge in $[y,z]$. Since the labels on the chain $\{u_i\}$ are assumed to be increasing, we must have $\gamma(u_0, u_1) = m_1$. It follows that $w_1 \vee^z u_0 \geq u_1$ and since $y \lessdot w_1$, we must have $u_1 = w_1$. Therefore, it suffices to consider the interval $[w_1, z]$. Thus, by induction on s, the two chains coincide. We conclude that the induced left modular maximal chain is the only chain in $[y, z]$ with increasing labels.

It also has the lexicographically least set of labels. To see this, suppose that $y = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_s = z$ is another chain in $[y, z]$. We assume that $u_1 \neq w_1$ since, otherwise, we can just restrict our attention to $[w_1, z]$. We have $\gamma(u_0, u_1) = m_l$, where $l = \min\{j \mid w_j \geq u_1\} \geq 2$ since $w_1 \not\geq u_1$. Hence $\gamma(u_0, u_1) \geq m_2 > \gamma(w_0, w_1)$. This gives that γ is an EL-labelling. (That γ is an EL-labelling was already shown in the lattice case in [21] and [44].)

Finally, we show that it is an interpolating EL-labelling. If $y \lessdot u \lessdot z$ is not the induced left modular maximal chain in $[y, z]$, then let $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$ be the induced left modular maximal chain. We will calculate $\gamma(y, u)$ and $\gamma(u, z)$ by restricting to $[y, z]$. We have that $\gamma(y, u) = m_l$ where $l = \min\{j \mid w_j \vee^z y \geq u\} = \min\{j \mid w_j \geq u\} = r$ since $u \lessdot z$. Therefore, $\gamma(y, u) = m_r$. Also, $\gamma(u, z) = m_l$ where $l = \max\{j + 1 \mid w_j \wedge_y z \leq u\} = \max\{j + 1 \mid w_j \leq u\} = 1$ since $y \lessdot u$. Therefore, $\gamma(y, u) = m_1$, as required. □

4.4 Left modularity from interpolating labellings

Our main aim for this section is to prove Theorem 4.2.7. We suppose that P is a bounded poset with an interpolating EL-labelling γ. Let $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ be the increasing chain from $\hat{0}$ to $\hat{1}$ and let $l_i = \gamma(x_{i-1}, x_i)$. We will begin by establishing some basic facts about interpolating labellings.

Let $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$. Suppose that, for some i, we have $\gamma(w_{i-1}, w_i) > \gamma(w_i, w_{i+1})$. Then the "basic replacement" at i takes the given chain and replaces the subchain $w_{i-1} \lessdot w_i \lessdot w_{i+1}$ with the increasing chain from w_{i-1} to w_{i+1}. In other words, the basic replacement at i is the analogue for posets that need not be graded of the action of U_i, as defined in Section 2.5. The basic tool for dealing with interpolating labellings is the following well-known fact about EL-labellings.

Lemma 4.4.1. *Let $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$. Successively perform basic replacements on this chain, and stop when no more basic replacements can be made. This algorithm terminates, and yields the increasing chain from y to z.*

Proof. At each step, the sequence of labels on the new chain lexicographically precedes the sequence on the old chain, so the process must terminate, and it is clear that it terminates in an increasing chain. □

We now prove some simple consequences of this lemma.

Lemma 4.4.2. *Let $\mathfrak{m}$ be the chain $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$. Then the labels on $\mathfrak{m}$ all occur on the increasing chain from y to z and are all different. Furthermore, all the labels on the increasing chain from y to z are bounded between the lowest and highest labels on $\mathfrak{m}$.*

Proof. That the labels on the given chain all occur on the increasing chain follows immediately from Lemma 4.4.1 and the fact that after a basic replacement, the labels on the old chain all occur on the new chain. Similar reasoning implies that the labels on the increasing chain are bounded between the lowest and highest labels on $\mathfrak{m}$.

That the labels are all different again follows from Lemma 4.4.1. Suppose otherwise. By repeated basic replacements, one obtains a chain which has two successive equal labels, which is not permitted by the definition of an interpolating labelling. $\square$

Lemma 4.4.3. *Let $z \in P$ such that there is some chain from $\hat{0}$ to z all of whose labels are in $\{l_1, \ldots, l_i\}$. Then $z \leq x_i$. Conversely, if $z \leq x_i$, then all the labels on any chain from $\hat{0}$ to z are in $\{l_1, \ldots, l_i\}$.*

Proof. We begin by proving the first statement. By Lemma 4.4.2, the labels on the increasing chain from $\hat{0}$ to z are in $\{l_1, \ldots, l_i\}$. Find the increasing chain from z to $\hat{1}$. Let w be the element in that chain such that all the labels below w on the chain are in $\{l_1, \ldots, l_i\}$, and those above it are in $\{l_{i+1}, \ldots, l_n\}$. Again, by Lemma 4.4.2, the increasing chain from $\hat{0}$ to w has all its labels in $\{l_1, \ldots, l_i\}$, and the increasing chain from w to $\hat{1}$ has all its labels in $\{l_{i+1}, \ldots, l_n\}$. Thus w is on the increasing chain from $\hat{0}$ to $\hat{1}$, and so $w = x_i$. But by construction $w \geq z$. So $x_i \geq z$.

To prove the converse, observe that by Lemma 4.4.2, no label can occur more than once on any chain. But since every label in $\{l_{i+1}, \ldots, l_n\}$ occurs on the increasing chain from x_i to $\hat{1}$, no label from among that set can occur on any edge below x_i. $\square$

The obvious dual of Lemma 4.4.3 is proved similarly:

Corollary 4.4.4. *Let $z \in P$ such that there is some chain from z to $\hat{1}$ all of whose labels are in $\{l_{i+1}, \ldots, l_n\}$. Then $z \geq x_i$. Conversely, if $z \geq x_i$, then all the labels on any chain from z to $\hat{1}$ are in $\{l_{i+1}, \ldots, l_n\}$.*

We are now ready to prove the necessary viability properties.

Lemma 4.4.5. *$x_i \vee z$ and $x_i \wedge z$ are well-defined for any $z \in P$ and for $i = 1, 2, \ldots, n$.*

Proof. We will prove that $x_i \wedge z$ is well-defined. The proof that $x_i \vee z$ is well-defined is similar. Let w be the maximum element on the increasing chain from $\hat{0}$ to z such that all labels on the increasing chain between $\hat{0}$ and w are in $\{l_1, \ldots, l_i\}$. Clearly $w \leq z$ and, by Lemma 4.4.3, $w \leq x_i$.

Suppose $y \leq z, x_i$. It follows that all labels from $\hat{0}$ to y are in $\{l_1, \ldots, l_i\}$. Consider the increasing chain from y to z. There exists an element u on this chain such that all the labels on the increasing chain from $\hat{0}$ to u are in $\{l_1, \ldots, l_i\}$ and all the labels on the increasing chain from u to z are in $\{l_{i+1}, \ldots, l_n\}$. Therefore, u is on the increasing chain from $\hat{0}$ to z and, in fact, $u = w$. Also, we have that $\hat{0} \leq y \leq u = w \leq z$. We conclude that w is the greatest common lower bound for z and x_i. $\square$

Lemma 4.4.6. *$\hat{0} = x_0 \wedge z \leq x_1 \wedge z \leq \cdots \leq x_n \wedge z = z$, after we delete repeated elements, is the increasing chain in $[\hat{0}, z]$. Hence, $(x_i \wedge z) \vee^z y$ is well-defined for $y \leq z$. Similarly, $(x_i \vee y) \wedge_y z$ is well-defined.*

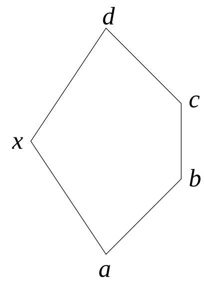


Figure 4-3: Relative ordering of a, b, c, d and x

Proof. From the previous proof, we know that $x_i \wedge z$ is the maximum element on the increasing chain from $\hat{0}$ to z such that all labels on the increasing chain between $\hat{0}$ and $x_i \wedge z$ are in $\{l_1, \ldots, l_i\}$. In particular, $x_i \wedge z \leq x_{i+1} \wedge z$ and so the first assertion holds.

Now apply Lemma 4.4.5 to the bounded poset $[\hat{0}, z]$, since it has an obvious interpolating labelling induced from the interpolating labelling of P. Recall that our definition of the existence of $(x_i \wedge z) \vee^z y$ only requires it to be well-defined in $[\hat{0}, z]$. The result follows. ☐

We conclude that the increasing maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ of P is viable. It remains to show that it is left modular.

Proof of Theorem 4.2.7. Suppose that x_i is not left modular for some i. Then there exists some pair $y \leq z$ such that $(x_i \vee y) \wedge_y z > (x_i \wedge z) \vee^z y$. Set $x = x_i$, $b = (x_i \wedge z) \vee^z y$ and $c = (x_i \vee y) \wedge_y z$. Observe that $d := x \vee b \geq x \vee y \geq c$ while $a := x \wedge c \leq x \wedge z \leq b$. So the picture is as shown in Figure 4-3.

By Lemma 4.4.3, the labels on the increasing chain from $\hat{0}$ to a are less than or equal to l_i. Consider the increasing chain from a to c. Let w be the first element along the chain. If $\gamma(a, w) \leq l_i$, then by Lemma 4.4.3, $w \leq x_i$, contradicting the fact that $a = x \wedge c$. Thus the labels on the increasing chain from a to c are all greater than l_i. Dually, the labels on the increasing chain from b to d are less than or equal to l_i. But now, by Lemma 4.4.2, the labels on the increasing chain from b to c must be contained in the labels on the increasing chain from a to c, and also from b to d. But there are no such labels, implying a contradiction. We conclude that the x_i are all left modular. ☐

We have shown that if P is a bounded poset with an interpolating labelling γ, then the unique increasing maximal chain M is a viable left modular maximal chain. By Theorem 4.2.6, M then induces an interpolating EL-labelling of P. We now show that this labelling agrees with γ for a suitable choice of label set. This is a special case of the following generalization of Lemma 2.3.3.

Proposition 4.4.7. *Let γ and δ be two interpolating EL-labellings of a bounded poset P. If γ and δ agree on the γ-increasing chain from $\hat{0}$ to $\hat{1}$, then γ and δ coincide.*

The proof is exactly analogous to the proof of Lemma 2.3.3 and is therefore omitted.

4.5 Generalizing supersolvability

Referring to Theorem 4.1.4, we see that we have successfully generalized the equivalence of properties (2) and (4) to arbitrary finite bounded posets. Naturally, we would like to generalize lattice supersolvability in a similar fashion. More precisely, suppose P is a bounded poset. For now, we consider the case of P being graded of rank n. We would like to define what it means for P to be *supersolvable*, thus generalizing Stanley's definition of lattice supersolvability. A definition of poset supersolvability with a different purpose appears in [44] but we would like a more general definition. In particular, we would like P to be supersolvable if and only if P has an S_n EL-labelling. For example, the poset shown in Figure 4-2, while it doesn't satisfy Welker's definition, should satisfy our definition. We need to define, in the poset case, the equivalent of a sublattice generated by two chains.

Suppose P has a viable maximal chain M. Thus $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ are well-defined for $x \in M$ and $y \leq z$ in P. Given any chain $\mathfrak{c}$ of P, we define $R_M(\mathfrak{c})$ to be the smallest subposet of P satisfying the following two conditions:

(i) M and $\mathfrak{c}$ are contained in $R_M(\mathfrak{c})$,

(ii) If $y \leq z$ in P and y and z are in $R_M(\mathfrak{c})$, then so are $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ for any x in M.

Definition 4.5.1. We say that a finite bounded poset P is *supersolvable* with M-chain M if M is a viable maximal chain and $R_M(\mathfrak{c})$ is a distributive lattice for any chain $\mathfrak{c}$ of P.

Since distributive lattices are graded, it is clear that a poset must be graded in order to be supersolvable. We now come to the main result of this section.

Theorem 4.5.2. *Let P be a finite bounded graded poset of rank n. Then the following are equivalent:*

1. *P has an S_n EL-labelling,*

2. *P has a viable left modular maximal chain,*

3. *P is supersolvable.*

Proof. Theorems 4.2.6 and 4.2.7 restricted to the graded case give us that (1) $\Leftrightarrow$ (2).

Our next step is to show that (1) and (2) together imply (3). Suppose P is a bounded graded poset of rank n with an S_n EL-labelling. Let M denote the increasing maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ of P. We also know that M is viable and left modular and, by Lemma 2.3.3, induces the same S_n EL-labelling. As in Section 2.6, given any maximal chain $\mathfrak{m}$ of P, we define $Q_{\mathfrak{m}}$ to be the closure of $\mathfrak{m}$ in P under basic replacements. In other words, $Q_{\mathfrak{m}}$ is the smallest subposet of P which contains M and $\mathfrak{m}$ and which has the property that, if y and z are in $Q_{\mathfrak{m}}$ with $y \leq z$, then the increasing chain between y and z is also in $Q_{\mathfrak{m}}$. By Proposition 2.6.2, we know that $Q_{\mathfrak{m}}$ is a distributive lattice. Now consider $R_M(\mathfrak{c})$. We will show that there exists

	Graded	**Not necessarily graded**
Lattice	1. Supersolvable 2. S_n EL-labelling 3. Good $\mathcal{H}_n(0)$ action 4. Left modular maximal chain	1. ? 2. Interpolating EL-labelling 3. ? 4. Left modular maximal chain
Not nec. Lattice	1. Supersolvable 2. S_n EL-labelling 3. Good $\mathcal{H}_n(0)$ action (in bowtie-free case) 4. Viable left mod. max. chain	1. ? 2. Interpolating EL-labelling 3. ? 4. Viable left mod. max. chain

Table 4.1: Equivalent properties

a maximal chain $\mathfrak{m}$ of P such that $R_M(\mathfrak{c}) = Q_\mathfrak{m}$. Let $\mathfrak{m}$ be the maximal chain of P which contains $\mathfrak{c}$ and which has increasing labels between successive elements of $\mathfrak{c} \cup \{\hat{0}, \hat{1}\}$. The only idea we need is that, for $y \leq z$ in P, the increasing chain from y to z is given by $y = (x_0 \vee y) \wedge_y z \leq (x_1 \vee y) \wedge_y z \leq \cdots \leq (x_n \vee y) \wedge_y z = z$, where we delete repeated elements. This follows from Lemma 4.3.4 since the induced left modular chain in $[y, z]$ has increasing labels. It now follows that $R_M(\mathfrak{c}) = Q_\mathfrak{m}$, and hence $R_M(\mathfrak{c})$ is a distributive lattice.

Finally, we will show that (3) $\Rightarrow$ (2). We suppose that P is a bounded supersolvable poset with M-chain M. Suppose $y \leq z$ in P and let $\mathfrak{c}$ be the chain $y \leq z$. For any x in M, $x \vee y$ is well-defined in P (because M is assumed to be viable) and equals the usual join of x and y in the lattice $R_M(\mathfrak{c})$. The same idea applies to $x \wedge z$, $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$. Since $R_M(\mathfrak{c})$ is a distributive lattice, we have that

$$(x \vee y) \wedge_y z = (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) = (x \wedge z) \vee y = (x \wedge z) \vee^z y$$

in $R_M(\mathfrak{c})$ and so M is left modular in P. □

Note. We know from Theorem 2.4.9 that a graded lattice of rank n is supersolvable if and only if it has an S_n EL-labelling. Therefore, it follows from Theorem 4.5.2 that the definition of a supersolvable poset restricts to graded lattices to give the usual definition. However, suppose P is a graded lattice with maximal chain M. We should note that it is not obvious, and may not even be true, that for a given chain $\mathfrak{c}$ of P, $R_M(\mathfrak{c})$ equals the sublattice of P generated by M and $\mathfrak{c}$.

We can informally summarize the main results of the last three chapters in the Table 4.1. This suggests the following questions.

Question 4.5.3. If P has an interpolating labelling, we can always define $Q_\mathfrak{m}$ to be the closure of $\mathfrak{m}$ under basic replacements. The argument for the equality of $R_M(\mathfrak{c})$ and $Q_\mathfrak{m}$ in the proof of Theorem 4.5.2 above holds even if P is not graded. However, in the ungraded case, it is certainly not true that $Q_\mathfrak{m}$ is distributive. The search for a full generalization of Theorem 4.1.4 thus leads us to ask what can be said about $Q_\mathfrak{m}$ in the ungraded case. Is it even a lattice? Can we say anything even in the case that P

is a lattice? Our overall goal with these questions would be to define supersolvability in a reasonable way for posets that need not be graded.

Question 4.5.4. Our work of Chapter 3 on good $\mathcal{H}_n(0)$ actions relied on our poset P being graded. If P is not graded, we can still consider the action on the maximal chains of P resulting from basic replacements. Does this action have any interesting properties?

Chapter 5

P-partitions and quasisymmetric functions

We now leave the general topic of edge labellings of a poset and move on to an exposition of the results of a different research project. While essentially separate from the material of Chapters 2, 3 and 4, this work is very similar in nature. For instance, it is also concerned with a distinction among the edges of a poset. A connection in terms of quasisymmetric functions is given by Proposition 5.1.2.

5.1 Stanley's (P, ω)-partitions conjecture

As with edge labellings of posets, the material of this chapter has its beginnings in [35], where Conjecture 5.1.5 below first appeared. First, we give the necessary definitions. Let P be a finite poset with n elements and let $\omega : P \to [n]$ be a bijection labelling the elements of P. We will often refer to elements of P by their images under ω. Unlike linear extensions of P as defined in Example 2.2.3, ω need not be order-preserving. The following definition first appeared in [35].

Definition 5.1.1. A (P, ω)-partition is a map $\sigma : P \to \mathbb{P}$ with the following properties:

(i) If $s \leq t$ in P then $\sigma(s) \leq \sigma(t)$. i.e. σ is order-preserving.

(ii) If $s \leq t$ and $\omega(s) > \omega(t)$ then $\sigma(s) < \sigma(t)$.

Thus a (P, ω)-partition is an order-preserving map from P to the positive integers with additional strictness conditions depending on ω. If $s \lessdot t$ is an edge of P and $\omega(s) > \omega(t)$, then we will refer to (s, t) as a *strict* edge. Otherwise, we will say that (s, t) is a *weak* edge. In particular, if ω is order-preserving, then all edges are weak and any order-preserving map from P to $\mathbb{P}$ is a P-partition. For more information on (P, ω)-partitions, see [14], [38, §4.5] and [41, §7.19]. We will denote the set of (P, ω)-partitions by $\mathcal{A}(P, \omega)$.

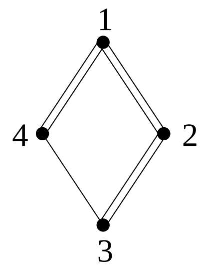


Figure 5-1: A poset P with its labelling ω

Our main object of study will be the (P,ω)-partition generating function $K_{P,\omega}(x)$ in the variables $x=(x_1,x_2,\ldots)$ defined by

$$K_{P,\omega}(x)=\sum_{\sigma\in\mathcal{A}(P,\omega)}\prod_{t\in P}x_{\sigma(t)}=\sum_{\sigma\in\mathcal{A}(P,\omega)}x_1^{\#\sigma^{-1}(1)}x_2^{\#\sigma^{-1}(2)}\cdots.$$

We see that $K_{P,\omega}(x)$ is a quasisymmetric function. In fact, the following result makes explicit a close connection between $K_{P,\omega}(x)$ and Ehrenborg's flag function $F_P(x)$ of Equation (3.1).

Proposition 5.1.2. *Suppose that ω is a linear extension. Then*

$$K_{P,\omega}(x)=F_{J(P)}(x).$$

Proof. Suppose σ is a (P,ω)-partition whose image in $\mathbb{P}$ has maximum value k. Then σ determines a multichain

$$\emptyset\subseteq t_1\subseteq t_2\subseteq\cdots\subseteq t_{k-1}\subset t_k=P \tag{5.1}$$

of order ideals of P, where $t_i=\{y\in P:\sigma(y)\leq i\}$. In fact, we see that this map from (P,ω)-partitions to multichains of $J(P)$ is a bijection. Also, the contribution of σ to $K_{P,\omega}(x)$ is

$$x_1^{|t_1|}x_2^{|t_2-t_1|}\cdots x_k^{|t_k-t_{k-1}|},$$

which is exactly the contribution of the multichain (5.1) to $F_{J(P)}(x)$. We conclude the result. $\square$

We now give some examples of (P,ω)-partitions.

Example 5.1.3. Suppose (P,ω) is given by Figure 5-1, where the double edges correspond to strict edge of P. We see that a (P,ω)-partition σ must fall into exactly one of the classes shown in Table 5.1. For simplicity, we write the quasisymmetric function $M_{\tau,n}$ defined by Equation 3.2 as $M_{\tau_1\cdots\tau_k}$, where $\tau=(\tau_1,\ldots,\tau_k)$ is a composition of n. We conclude that $K_{P,\omega}(x)=M_{211}+M_{121}+2M_{1111}$. Therefore, the monomial $x_1^2x_2x_3$ appears with coefficient 1 in $K_{P,\omega}(x)$ whereas $x_1x_2x_3^2$ has coefficient 0. In particular, $K_{P,\omega}(x)$ is not symmetric. In general, suppose that we have a quasisymmetric function $f=\sum c_\alpha M_\alpha$, where the sum is over all compositions α of $n\in\mathbb{P}$.

Values of σ	Contribution to $K_{P,\omega}(x)$
$\sigma(3) = \sigma(4) < \sigma(2) < \sigma(1)$	M_{211}
$\sigma(3) < \sigma(4) = \sigma(2) < \sigma(1)$	M_{121}
$\sigma(3) < \sigma(4) < \sigma(2) < \sigma(1)$	M_{1111}
$\sigma(3) < \sigma(2) < \sigma(4) < \sigma(1)$	M_{1111}

Table 5.1: (P, ω)-partitions for Figure 5-1

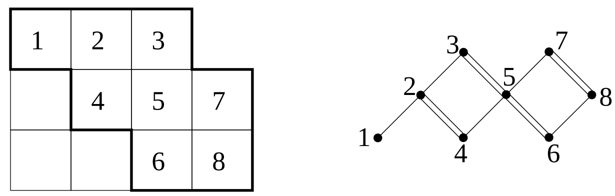


Figure 5-2: A Schur labelled skew shape and its corresponding labelled poset

We see that f is a symmetric function if and only if $c_\alpha = c_\beta$ whenever α and β are compositions with the same multiset of parts.

Example 5.1.4. We let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a *partition* of the number n. (i.e. $\lambda_i \in \mathbb{N}$, $\lambda_i \geq \lambda_{i+1}$ and $\sum_i \lambda_i = n$.) We draw the Young diagram of λ in French notation. For example, the boxes on the left in Figure 5-2 show the partition $\lambda = (4, 4, 3)$. We identify λ with the sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k, 0, 0, \ldots)$. If μ is another partition of n then we say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. This is equivalent to saying that the diagram of μ is contained in the diagram of λ. If $\mu \subseteq \lambda$, then we define the *skew shape* λ/μ to be the set of boxes in the diagram of λ that remain after we remove those boxes corresponding to the partition μ. For example, the boxes of Figure 5-2 surrounded by a heavy line correspond to the shape $(4, 4, 3)/(2, 1)$.

If λ/μ has n boxes, written $|\lambda/\mu| = n$, then we define a *Schur labelling* of λ/μ to be a labelling of the boxes of λ/μ with the numbers $[n]$ that increases down columns and from left to right along rows. Given a Schur labelling ω of λ/μ, let $(P_{\lambda/\mu}, \omega)$ denote the labelled poset suggested by rotating the boxes of λ/μ by 45° counterclockwise. All of these definitions are best explained by an example and Figure 5-2 shows a Schur labelling ω of λ/μ and the corresponding labelled poset $(P_{\lambda/\mu}, \omega)$. We say that $(P_{\lambda/\mu}, \omega)$ is a *Schur labelled skew shape poset* or just a *skew shape poset*.

We see that a (P, ω)-partition of a skew shape poset $(P_{\lambda/\mu}, \omega)$ corresponds to an assignment of positive integers to the boxes of λ/μ that weakly increases from left to right along rows and strictly increases up columns. This is exactly the definition of a *semistandard Young tableau* of shape λ/μ. Furthermore, the quasisymmetric function $K_{P_{\lambda/\mu},\omega}$ gives us exactly the *Schur function* $s_{\lambda/\mu}$. A nice combinatorial proof from [1] that $s_{\lambda/\mu}$ is symmetric also appears as [41, Theorem 7.10.2]. We do not include it here as we use the same argument to prove the slightly more general Theorem 5.4.5. We conclude that $K_{P,\omega}(x)$ is symmetric if (P, ω) is a skew shape poset.

This brings us to Stanley's P-partitions Conjecture. We say that two labelled posets are isomorphic if there exists a poset isomorphism between them that sends

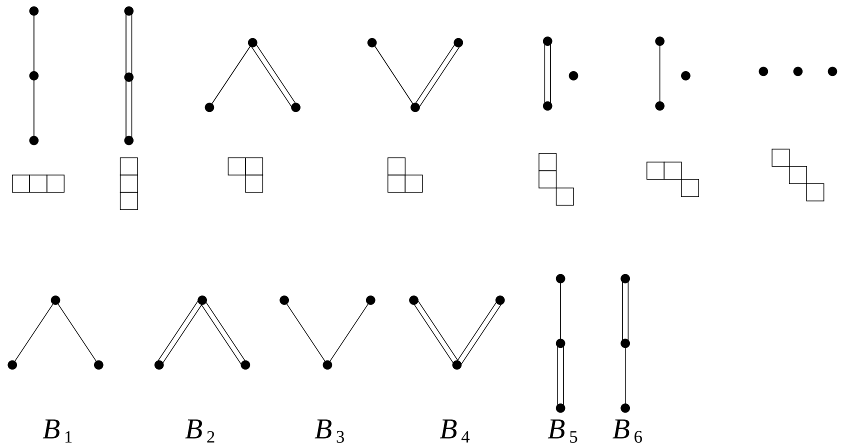


Figure 5-3: The 13 labelled posets with 3 elements

weak edges to weak edges and strict edges to strict edges.

Conjecture 5.1.5. *Let (P,ω) be a labelled poset. $K_{P,\omega}(x)$ is symmetric if and only if (P,ω) is isomorphic to a Schur labelled skew shape poset.*

In [38, Exercise 4.23] and [42], this conjecture is shown to be true when ω is a linear extension. John Stembridge has verified the conjecture for all posets P with $|P| \leq 7$.

To prove the conjecture in this form, we would have the daunting task of deducing information about the global structure of (P,ω) from the symmetry of $K_{P,\omega}(x)$. A reformulation in terms of local conditions on (P,ω) is the subject of the next section.

5.2 Malvenuto's reformulation

This subject of this section is the work of Claudia Malvenuto on Stanley's (P,ω)-partitions conjecture. The proofs can be found in [24], with some clarification and further analysis of the implications of her results in [25].

The reader may already have observed that to calculate $K_{P,\omega}(x)$, we don't need to know the full labelling ω. It suffices to know which edges are strict and which edges are weak. Therefore, from now on, we will often omit the labels on the vertices, and when we refer to a "labelled poset," we mean a poset with strict and weak edges which come from some underlying labelling.

Consider the set of all possible labelled posets with 3 elements, as shown in Figure 5-3. The posets in the first row are shown with a corresponding skew shape. In particular, all the posets in the first row are skew shape posets and so have symmetric generating functions. On the other hand, we can see that none of the posets in the second row are skew shape posets. Also, none of their generating functions, which are shown in Table 5.2, are symmetric. We will refer to these six posets as "forbidden" posets.

A subposet Q of a poset P is said to be convex if $b \in Q$ whenever $a, c \in Q$ with $a \leq b \leq c$ in P. From the remarks above, we know that a skew shape poset $(P_{\lambda/\mu},\omega)$

(P,ω)	$K_{P,\omega}(x)$
B_1	$M_3 + M_{21} + 2M_{12} + 2M_{111}$
B_2	$M_{21} + 2M_{111}$
B_3	$M_3 + 2M_{21} + M_{12} + 2M_{111}$
B_4	$M_{12} + 2M_{111}$
B_5	$M_{12} + M_{111}$
B_6	$M_{21} + M_{111}$

Table 5.2: Generating functions for the 6 forbidden posets

cannot have any of the forbidden posets as a convex subposet. We say that $(P_{\lambda/\mu}, \omega)$ "has no forbidden convex subposets." Malvenuto proved the following converse result.

Theorem 5.2.1. *Let (P,ω) be a labelled poset. If (P,ω) has no forbidden convex subposets, then (P,ω) is isomorphic to a skew shape poset.*

We conclude that a labelled poset (P,ω) is isomorphic to a skew shape poset if and only if it has no forbidden convex subposets. In particular, if (P,ω) has no forbidden convex subposets, then it is isomorphic to a skew shape poset and hence $K_{P,\omega}(x)$ is symmetric. It follows that Stanley's conjecture reduces to the following statement.

Conjecture 5.2.2. *Let (P,ω) be a labelled poset. If $K_{P,\omega}(x)$ is symmetric then (P,ω) has no forbidden convex subposets.*

We will discuss our progress with this conjecture in Section 5.6.

Remark 5.2.3. We can also formulate a statement of the conjecture that doesn't refer to forbidden posets or to skew shape posets. Suppose (P,ω) is a labelled poset and Q is a convex subposet of P. We will use $\omega|_Q$ to denote the labelling ω restricted to the elements of Q. If (P,ω) has no forbidden convex subposets, then clearly $(Q,\omega|_Q)$ has no forbidden convex subposets and by Theorem 5.2.1, $K_{Q,\omega|_Q}$ is symmetric. We also know that if $K_{Q,\omega|_Q}$ is symmetric, then $(Q,\omega|_Q)$ can not be one of the forbidden posets. Hence, (P,ω) has no forbidden convex subposets if and only if $K_{Q,\omega|_Q}$ is symmetric for all convex subposets Q of P. Therefore, showing Stanley's conjecture is equivalent to showing that if $K_{P,\omega}(x)$ is symmetric then so is $K_{Q,\omega|_Q}$ for every convex subposet Q of P. By induction on $|P|-|Q|$, it even suffices to show this just for $|Q| = |P|-1$. However, we found this formulation to be far more difficult to work with than Conjecture 5.2.2.

5.3 Oriented Posets

As we observed, given a labelled poset (P,ω), the generating function $K_{P,\omega}(x)$ depends only on the poset P and the designation of edges as strict or weak. This suggests that, given a poset P, we might choose any designation O of strict and weak edges. We define a (P,O)-partition in the obvious manner:

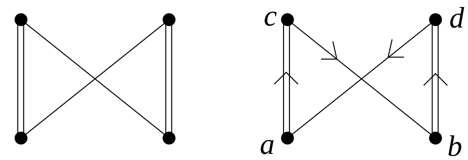


Figure 5-4: Orienting the edges of a poset

Definition 5.3.1. Let P be a poset with a designation O of strict and weak edges. A (P, O)-partition is a map $\sigma : P \to \mathbb{P}$ with the following properties:

(i) If $s \leq t$ in P then $\sigma(s) \leq \sigma(t)$. i.e. σ is order-preserving.

(ii) If $s \leq t$ and (s, t) is a strict edge, then $\sigma(s) < \sigma(t)$.

We denote the set of (P, O)-partitions by $\mathcal{A}(P, O)$.

For example, consider the poset (P, O) shown on the left in Figure 5-4. We orient the strict edges of the poset upward and the weak edges of the poset downward. Therefore, if the designation of strict and weak edges came from a labelling ω of P, then the arrow always points to the smaller label. (One can think of the arrow as begin a less-than sign which compares the labels at the end of the edge.) In certain settings, it will helpful to use this idea of thinking of the designation of strict and weak edges as an *orientation* of the Hasse diagram of P. This explains the use of the letter "O" for the designation of strict and weak edges. We will refer to (P, O) as an *oriented* poset. Occasionally, we will think of (P, O) as a directed graph, with the direction on the edges coming from the orientation O (and not from the partial ordering of P).

A *walk* in (P, O) is a walk in the directed graph (P, O). In particular, a walk must follow the direction of the arrows. A *cycle* of (P, O) is a closed walk and we say that (P, O) is *acyclic* if and only if it has no cycles. For example, the oriented poset in Figure 5-4 contains exactly one cycle, namely the closed walk which, say, starts at a, then goes in turn to c, b and d before returning to a. We write this cycle as $a \to c \to b \to d \to a$ or, alternatively, with b, c or d as the starting and finishing point. It is clear that the designation of strict and weak edges in this example cannot come from a labelling ω. Indeed, suppose that (P, O) actually corresponds to a labelled poset (P, ω). Then ω would have to satisfy $\omega(a) > \omega(c) > \omega(b) > \omega(d) > \omega(a)$, which is impossible. In general, we can use the same argument to show that if any oriented poset has a cycle, then its designation of strict and weak edges cannot come from a labelling ω of the vertices. In fact, we have the appropriate converse result, as stated in the following lemma.

Lemma 5.3.2. *Let (P, O) be an oriented poset. The designation of strict and weak edges of (P, O) can come from a labelling of the vertices of P if and only if (P, O) is acyclic.*

Proof. If (P, O) contains a cycle, then by the argument above, the designation of strict and weak edges cannot come from a labelling.

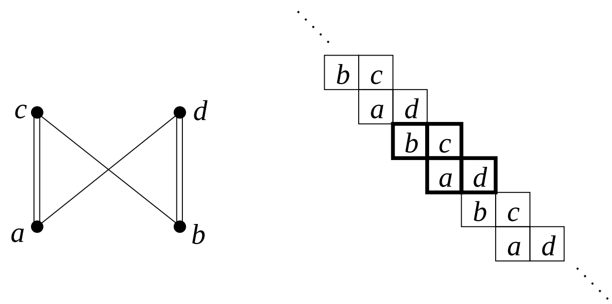


Figure 5-5: Generalizing the idea of a skew shape poset

Now suppose that (P, O) is acyclic. Define a new ordering $\leq_Q$ on the elements of P by $x \leq_Q y$ if and only if there is a walk in (P, O) from y to x. It is readily checked that $\leq_Q$ is a partial ordering. Let Q be the resulting poset, and let ω be any linear extension of Q. Since Q has the same underlying set as P, ω can also be viewed as a labelling of the vertices of P. If (x, y) is a strict edge in (P, O), then there is a walk from x to y, so $x \geq_Q y$ and hence $\omega(x) \geq \omega(y)$. Similarly, if (x, y) is a weak edge in (P, O) then $\omega(x) \leq \omega(y)$. We conclude that the labelled poset (P, ω) has the required designation of strict and weak edges. □

We define the generating function $K_{P,O}(x)$ analogously to $K_{P,\omega}(x)$:

$$K_{P,O}(x) = \sum_{\sigma \in \mathcal{A}(P,O)} x_1^{\#\sigma^{-1}(1)} x_2^{\#\sigma^{-1}(2)} \cdots .$$

Example 5.3.3. Consider again the oriented poset (P, O) shown in Figure 5-4. Using the same method as in Example 5.1.3, we can compute that

$$K_{P,O}(x) = M_{22} + 2M_{211} + 2M_{121} + 2M_{112} + 4M_{1111}.$$

Notice that $K_{P,O}(x)$ is symmetric and also that (P, O) has no forbidden convex subposets. So, even though (P, O) is not a labelled poset, it is still consistent with Conjecture 5.2.2. Next, it is natural to ask if (P, O) is isomorphic to a skew shape poset. However, we know that skew shape posets are all labelled posets since they come from Schur labelled skew shapes. Therefore, (P, O) cannot be a skew shape poset. In any case, suppose we try to construct a skew shape corresponding to (P, O). Referring now to Figure 5-5, we see that we need the box corresponding to a to be directly below the box corresponding to c and directly to the left of the box corresponding to d. Also, we need the box corresponding to b to be directly below the box corresponding to d and directly to the left of the box corresponding to c. Naively putting this all together, we might be led to the construction on the right in Figure 5-5. We refer to such constructions as *cylindric skew shapes*. The author first encountered them in the work of Alexander Postnikov [30], where they are called *cylindric diagrams*. This example motivates the formal definitions of the next section.

5.4 Cylindric skew shapes

Cylindric skew shapes will allow us to generalize Conjecture 5.1.5 and Theorem 5.2.1 to oriented posets. For the following introduction to the notation and definitions related to cylindric skew shapes, we largely follow [30]. We remark that related objects, known as *cylindric partitions*, which extend the idea of plane partitions, are studied in [15].

Fix integers $u, v \geq 2$. We define the *cylinder* $\mathfrak{C}_{uv}$ to be the following quotient of $\mathbb{Z}^2$:

$$\mathfrak{C}_{uv} = \mathbb{Z}^2/(-u, v)\mathbb{Z}.$$

In other words, $\mathfrak{C}_{uv}$ is the quotient of the integer lattice $\mathbb{Z}^2$ modulo a shifting action which sends (i, j) to $(i-u, j+v)$. For $(i, j) \in \mathbb{Z}^2$, we let $\langle i, j\rangle = (i, j) + (-u, v)\mathbb{Z}$ denote the corresponding element of $\mathfrak{C}_{uv}$. $\mathfrak{C}_{uv}$ inherits a natural partial order $\leq_{\mathfrak{C}}$ from $\mathbb{Z}^2$ which is defined by its covering relations $\langle i, j\rangle \lessdot_{\mathfrak{C}} \langle i+1, j\rangle$ and $\langle i, j\rangle \lessdot_{\mathfrak{C}} \langle i, j+1\rangle$.

Note. This partial order is antisymmetric since u and v are positive. We require $u, v \geq 2$ to ensure that $\langle i, j\rangle <_{\mathfrak{C}} \langle i+1, j\rangle$ and $\langle i, j\rangle <_{\mathfrak{C}} \langle i, j+1\rangle$ are covering relations. Indeed, suppose that $u = 1$ and $v > 1$. Then we would have

$$\langle 0, 0\rangle <_{\mathfrak{C}} \langle 0, 1\rangle <_{\mathfrak{C}} \cdots <_{\mathfrak{C}} \langle 0, v\rangle = \langle 1, 0\rangle$$

and so $\langle 0, 0\rangle$ is not covered by $\langle 1, 0\rangle$. We have a similar problem if $v = 1$. The covering relations $\langle i, j\rangle \lessdot_{\mathfrak{C}} \langle i+1, j\rangle$ and $\langle i, j\rangle \lessdot_{\mathfrak{C}} \langle i, j+1\rangle$ are essential in the definition below of cylindric skew shape posets.

Definition 5.4.1. A *cylindric skew shape* is a finite convex subposet of the poset $\mathfrak{C}_{uv}$.

Example 5.4.2. We can regard skew shapes λ/μ as a special case of cylindric skew shapes. Suppose λ/μ fits inside a box of width u and height v, where we always choose u and v to be at least 2. We embed λ/μ in $\mathfrak{C}_{uv}$ by mapping the box in the ith row and jth column of λ/μ to $\langle i, j\rangle$. Figure 5-6 shows the resulting image of λ/μ in $\mathbb{Z}^2$, with one representative of λ/μ shown in bold. Notice that elements of different representatives of λ/μ are always incomparable in $\mathbb{Z}^2$. Of course, we could also embed λ/μ in $\mathfrak{C}_{u'v'}$ where $u' \geq u$ and $v' \geq v$. This would result in extra space between the representatives of λ/μ in $\mathbb{Z}^2$. For example, setting $u' = u+1$ and $v' = v$ would leave a blank extra column of $\mathbb{Z}^2$ between any two neighboring representatives of λ/μ.

Example 5.4.3. The construction on the right in Figure 5-5 is a cylindric skew shape with $u = 2$ and $v = 2$,

Let C be any cylindric skew shape which is a subposet of the cylinder $\mathfrak{C}_{uv}$. The elements of C inherit a partial order from $\mathfrak{C}_{uv}$. Suppose we consider the vertical edges $\langle i, j\rangle \lessdot_{\mathfrak{C}} \langle i, j+1\rangle$ of C to be strict and the horizontal edges $\langle i, j\rangle \lessdot_{\mathfrak{C}} \langle i+1, j\rangle$ to be weak. This designation of strict and weak edges makes C into an oriented poset (P, O), which we refer to as a *cylindric skew shape poset.* For example, we encountered a cylindric skew shape poset in Figure 5-5. Also, because of Example 5.4.2, skew shape posets are always cylindric skew shape posets.

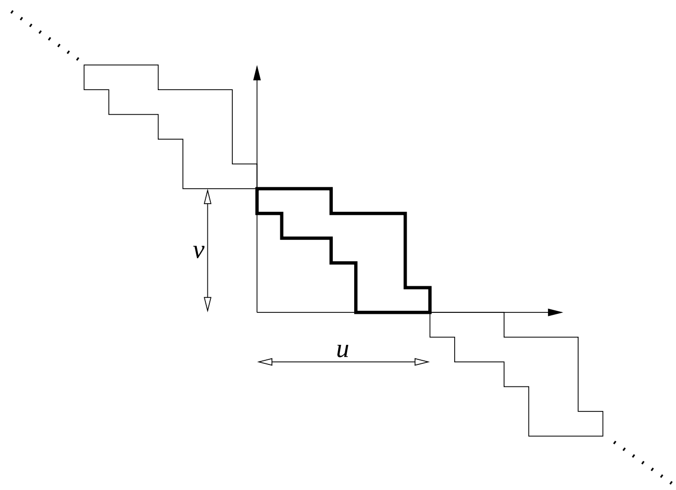


Figure 5-6: Skew shapes are cylindric skew shapes

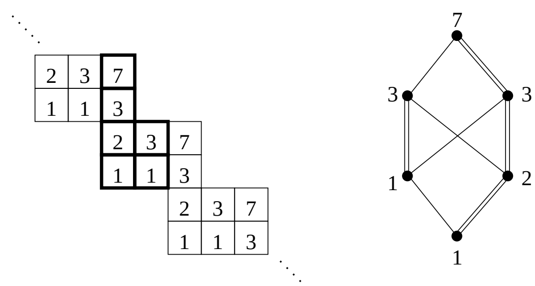


Figure 5-7: A cylindric tableau and its cylindric skew shape poset

Suppose C is a cylindric skew shape which is a subposet of the cylinder $\mathfrak{C}_{uv}$. Let us define what we mean by the rows and columns of C. The *p-th row* is the set $\{\langle i,j\rangle \in C \mid j = p\}$ and the *q-th column* is the set $\{\langle i,j\rangle \in C \mid i = q\}$. [1] So the rows only depend on $p \pmod v$ and the columns only depend on $q \pmod u$. Thus the cylinder $\mathfrak{C}_{uv}$ has exactly v rows and u columns.

Finally, suppose that (P,O) is a cylindric skew shape poset derived from a cylindric skew shape C. We see that a (P,O)-partition corresponds to an assignment σ of positive integers to the boxes of C that weakly increases from left to right along rows and strictly increases up columns. We call such an assignment a *semistandard cylindric tableau* of shape C. Semistandard cylindric tableaux appear in [2] under the name "proper tableaux."

Example 5.4.4. Figure 5-7 shows a semistandard cylindric tableau as well as the corresponding cylindric skew shape poset (P,O), with elements labelled by their images under the corresponding (P,O)-partition.

Theorem 5.4.5. *Suppose (P,O) is a cylindric skew shape poset derived from a cylindric skew shape C. Then $K_{P,O}(x)$ is symmetric.*

[1] In [30], rows and columns are defined the other way around.

Proof. For the sake of completeness, we reproduce here the proof from [41, Theorem 7.10.2], since the proof in the case of skew shape posets extends directly to the cylindric case.

Since every permutation is a composition of adjacent transpositions, it suffices to show that $K_{P,O}(x)$ is invariant under the interchanging of x_i and x_{i+1}. Suppose that $|C| = n$ and that $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a *weak* composition of n. (i.e. $\sum_i \alpha_i = n$ and $\alpha_i \in \mathbb{N}$.) Let

$$\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, \alpha_{i+3}, \ldots).$$

If $\mathcal{T}_{C,\alpha}$ denotes the set all semistandard cylindric tableau of shape C with α_i parts equal to i for all i, then we seek a bijection $\varphi\colon \mathcal{T}_{C,\alpha} \to \mathcal{T}_{C,\tilde{\alpha}}$.

Let $T \in \mathcal{T}_{C,\alpha}$. Consider the parts of T equal to i or $i+1$. Some columns of T will contain no such parts, while some others will contain both i and $i+1$. These column we ignore. The remaining parts equal to i or $i+1$ occur once in each column, and consist of rows with a certain number r of i's followed by a certain number s of $i+1$'s, where r and s depend on the row in question. For example, a portion of T could look as follows:

$$\begin{array}{ccccc} i+1 & i+1 & & & \\ i & i & \underbrace{i \quad i}_{r=2} & \underbrace{i+1 \quad i+1 \quad i+1 \quad i+1}_{s=4} & i+1 \\ & & & & i \end{array}$$

In each such row, convert the r i's and s $i+1$'s to s i's and r $i+1$'s:

$$\begin{array}{ccccc} i+1 & i+1 & & & \\ i & i & \underbrace{i \quad i \quad i \quad i}_{s=4} & \underbrace{i+1 \quad i+1}_{r=2} & i+1 \\ & & & & i \end{array}$$

We easily see that the resulting array $\varphi(T)$ belongs to $\mathcal{T}_{C,\tilde{\alpha}}$, and that φ establishes the desired bijection. □

We say that two oriented posets are isomorphic if there exists a poset isomorphism between them that sends weak edges to weak edges and strict edges to strict edges. Our investigations have suggested the following extension of Stanley's Conjecture 5.1.5.

Conjecture 5.4.6. *Let (P, O) be an oriented poset. $K_{P,O}(x)$ is symmetric if and only if every connected component of (P, O) is isomorphic to a cylindric skew shape poset.*

In order to investigate this conjecture, we would like an analogue of Malvenuto's Theorem 5.2.1. This will be the subject of the next section.

5.5 Extending the reformulation

Recall the forbidden convex subposets of Figure 5-3. For the same reason that skew shape posets have no forbidden convex subposets, cylindric skew shape posets also

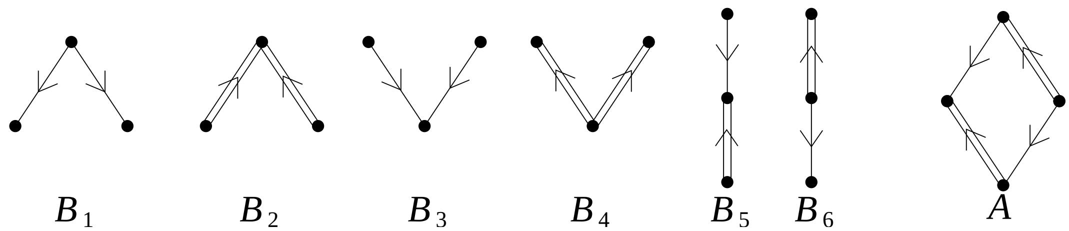


Figure 5-8: The six forbidden oriented convex subposets

have no forbidden convex subposets.

Our goal for this section is to reformulate Conjecture 5.4.6 by extending Theorem 5.2.1 as follows:

Theorem 5.5.1. *Let (P, O) be an oriented poset. If (P, O) has no forbidden convex subposets, then every connected component of (P, O) is isomorphic to a cylindric skew shape poset.*

We will need two lemmas about oriented posets, the second of which gives some insight into their structure. The first lemma appears in [24] and [25].

In Figure 5-8, we show the six forbidden convex subposets along with the appropriate orientation of their edges, followed by an allowed subposet A.

Lemma 5.5.2. *Suppose (P, O) is an oriented poset which has no forbidden convex subposets. If a subposet of the form B_5 or B_6 appears as a non-convex subposet of (P, O), then then it can only appear as part of an interval of the form A.*

Proof. Suppose, without loss of generality, that the subposet B_5 appears in (P, O). Label its elements by w, y and z where $w \lessdot y \lessdot z$. Since B_5 is not a convex subposet, there must be an element v of (P, O) such that $w < v < z$ and $v \neq y$, with a chain from w to v and another chain from v to z. Using the fact that (P, O) has no forbidden convex subposets, we can show that both of these chains have length one, that $w \lessdot v$ is a weak edge and that $v \lessdot z$ is a strict edge. We leave the details as a nice exercise for the reader that is helpful in building intuition for oriented posets with no forbidden convex subposets. The details can also be found in [24, Corollary 2] or [25, Corollaire 6.10]. □

Lemma 5.5.3. *Let (P, O) be a connected oriented poset which has no forbidden convex subposets. If (P, O) has a cycle, then every element of P is in a cycle.*

Proof. Suppose w is an element of P which is not in a cycle. Since P is finite, it suffices to consider the case when there exists an element z of P, in a cycle $\mathcal{C}$, which is connected to w by an edge of (P, O). We can assume that $z \lessdot w$ and that the edge (z, w) is strict. Indeed, if $z \lessdot w$ is a weak edge, then we can apply the argument below where we reverse the role of strict and weak edges. If $w \lessdot z$, then we can apply a similar argument to the dual poset P^*.

Since $z \in \mathcal{C}$, $w \notin \mathcal{C}$ and $z \lessdot w$ is a strict edge, we are limited in the possible configuration of edges of $\mathcal{C}$ that can occur at z. In fact, we see that the two configurations

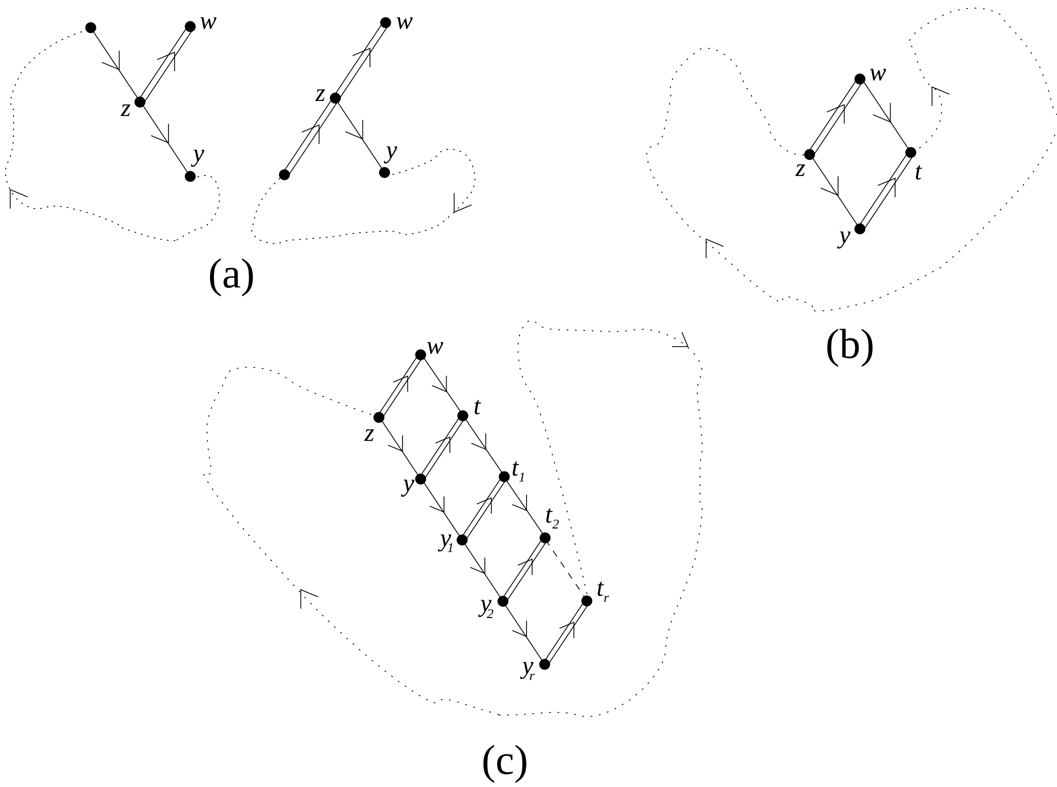


Figure 5-9: Figures for proof of Lemma 5.5.3

shown in Figure 5-9(a) are the only possibilities, where the dotted line represents the remainder of $\mathcal{C}$. In either case, there exists an element $y \in \mathcal{C}$ such that $y \lessdot z$ is a weak edge. Notice that the subposet $y \lessdot z \lessdot w$ is of the form B_6, so by Lemma 5.5.2, there must be a strict edge $y \lessdot t$ such that $t \lessdot w$ is a weak edge.

Case 1: After going from z to y, the next edge of $\mathcal{C}$ is a strict edge. Since this edge is strict, it must be of the form $y \lessdot t'$. Now since B_4 is a forbidden convex subposet, we must have that $t' = t$. See Figure 5-9(b). We see that we have a walk which starts at w, goes to t, follows $\mathcal{C}$ around to z and then returns to w. Therefore, w is in a cycle.

Case 2: After going from z to y, the next r edges of $\mathcal{C}$ are weak edges, where $r \geq 1$. Suppose this portion of $\mathcal{C}$ is of the from $z \to y \to y_1 \to y_2 \to \cdots \to y_r \to t_r$, where $y_r \lessdot t_r$ is a strict edge. Notice that the subposet $y_1 \lessdot y \lessdot t$ is of the form B_6, so by Lemma 5.5.2, there must be a strict edge $y_1 \lessdot t_1$ such that $t_1 \lessdot t$ is a weak edge. We continue this process as in Figure 5-9(c). Eventually, we construct a walk $w \to t \to t_1 \to t_2 \to \cdots \to t_r$ along weak edges. Now construct a cycle which starts at w, first follows this walk, then follows $\mathcal{C}$ around to z and finally returns to w. ☐

It is worth noting that the new cycles constructed in the proof above have the same number of edges, weak edges and strong edges as $\mathcal{C}$.

Proof of Theorem 5.5.1. Suppose (P, O) is an oriented poset which has no forbidden convex subposets. We wish to show that every connected component of (P, O) is isomorphic to a cylindric skew shape poset. Therefore, it suffices to assume that (P, O) is connected.

Case 1: (P, O) is acyclic. By Lemma 5.3.2, (P, O) is thus a labelled poset. So by Theorem 5.2.1, (P, O) is isomorphic to a skew shape poset. Hence, by Example 5.4.2, (P, O) is isomorphic to a cylindric skew shape poset.

Case 2: (P, O) has a cycle. Our proof uses several ideas from Malvenuto's proof of Theorem 5.2.1. We proceed by induction on $|P|$. Let m be a maximal element of P. By Lemma 5.5.3, m must be in a cycle $\mathcal{C}$. In particular, m must cover at least two elements of P. Since B_1 and B_2 are forbidden convex subposets, m covers exactly one element w of P with a weak edge and also covers exactly one element z of P with a strict edge. Clearly, $w, z \in \mathcal{C}$. Let Q be the subposet of P obtained from P by removing m. Let $O|_Q$ denote the orientation O restricted to the edges of Q, and consider the oriented poset $(Q, O|_Q)$. Now $(Q, O|_Q)$ must still be connected since $\mathcal{C}$ is still connected if we remove m and so there is a walk from w to z. Also, $(Q, O|_Q)$ cannot have any forbidden convex subposets. If $(Q, O|_Q)$ is acyclic then, by Case 1, it is isomorphic to a cylindric skew shape poset. If $(Q, O|_Q)$ has a cycle then, by our induction hypothesis, it is again isomorphic to a cylindric skew shape poset. Therefore, there exists a poset embedding $e : Q \to \mathfrak{C}_{uv}$ such that $C = e(Q)$ is a cylindric skew shape and such that C and $(Q, O|_Q)$ are isomorphic as oriented posets. For convenience, we will choose u and v to be as small as possible. We wish to extend the domain of e to P and define $e(m)$ so that $C \cup e(m)$ is a cylindric skew shape and is isomorphic to (P, O) as an oriented poset.

For any element q of Q we denote $e(q)$ by $\langle i_q, j_q \rangle$. Since $w \lessdot m$ is a weak edge and B_3 is a forbidden subposet, $\langle i_w + 1, j_w \rangle \notin C$. Similarly, $\langle i_z, j_z + 1 \rangle \notin C$. In fact, since C is a cylindric skew shape and is thus convex, $\langle i_w + r, j_w \rangle, \langle i_z, j_z + r \rangle \notin C$ for any $r \geq 1$. Suppose $\langle i_w, j_w - 1 \rangle$ is in C and equals $e(q)$ for $q \in Q$. Then $q \lessdot w$ is a strict edge in $(Q, O|_Q)$ so also in (P, O). Hence $q \lessdot w \lessdot m$ is a subposet of (P, O) the form B_5 and, by Lemma 5.5.2, there exists $z' \in P$ such that $q \lessdot z' \lessdot m$ and $z' \lessdot m$ is a strict edge. We must have that $z' = z$ since, otherwise, (P, O) would have a convex subposet of the form B_2. In particular, we get that $\langle i_w + 1, j_w - 1 \rangle = \langle i_z, j_z \rangle$, as in Figure 5-10(a). We set $e(m) = \langle i_z, j_w \rangle$. Since C is a convex subposet of $\mathfrak{C}_{uv}$, it is clear that $C \cup e(m)$ is still a convex subposet of $\mathfrak{C}_{uv}$ and hence is a cylindric skew shape. We also have that $C \cup e(m)$ and (P, O) are isomorphic as oriented posets, as required. Therefore, it remains to consider the case when $\langle i_w, j_w - 1 \rangle \notin C$.

If we have $\langle i_z - 1, j_z \rangle \in C$, then a similar argument will show again that $\langle i_w + 1, j_w - 1 \rangle = \langle i_z, j_z \rangle$ and we continue as before. So we can assume that $\langle i_w, j_w - 1 \rangle, \langle i_z - 1, j_z \rangle \notin C$. We are in the situation shown in Figure 5-10(b). Since there is no element of C directly below or directly to the right of $e(w)$, in $(Q, O|_Q)$ there is no $q \in Q$ such that $q \lessdot w$ is a strict edge or $w \lessdot q$ is a weak edge. In other words, there are no edges pointing in to w. Therefore w is not in a cycle, and so by Lemma 5.5.3, $(Q, O|_Q)$ is acyclic. Lemma 5.3.2 then tells us that $(Q, O|_Q)$ is a labelled poset and so, by Theorem 5.2.1, $(Q, O|_Q)$ is a skew shape poset. Since $(Q, O|_Q)$ is connected, C must be rook-wise connected as in Figure 5-10(c). Since we chose u and v to be minimal, we again have that $\langle i_w + 1, j_w - 1 \rangle = \langle i_z, j_z \rangle$. As before, we set $e(m) = \langle i_z, j_w \rangle$ and $C \cup e(m)$ has the required properties. $\square$

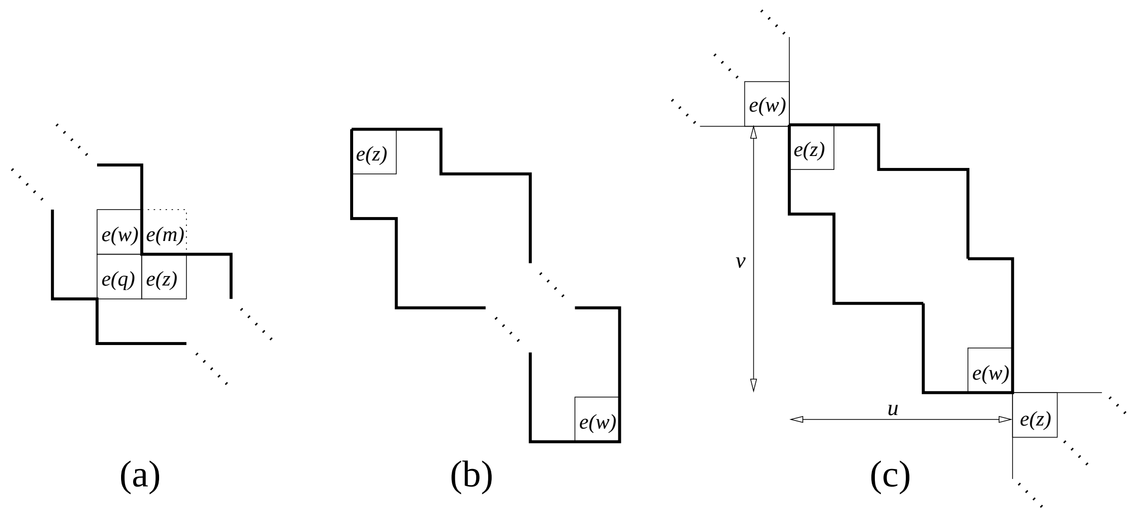


Figure 5-10: Figures for proof of Theorem 5.5.1

It follows that a connected oriented poset is a cylindrical skew shape poset if and only if it has no forbidden convex subposets. We can now restate Conjecture 5.4.6.

Conjecture 5.5.4. *Let (P, O) be an oriented poset. If $K_{P,O}(x)$ is symmetric then (P, O) has no forbidden convex subposets.*

As we know, Stanley's Conjecture 5.1.5 is a special case of this conjecture. Our approach will be to consider oriented posets (P, O) which have forbidden convex subposets and attempt to show that $K_{P,O}(x)$ is not symmetric. In the next section, we exhibit techniques which allow us to succeed in a large number of cases.

5.6 Special cases

We suppose that (P, O) is an oriented poset which contains a forbidden convex subposet B. Our underlying goal is to show that $K_{P,O}(x)$ is not symmetric. In Proposition 5.6.2, we prove that $K_{P,O}(x)$ is not symmetric if B is the only forbidden convex subposet of (P, O). In Proposition 5.6.3, we show that $K_{P,O}(x)$ is not symmetric if B is somehow "higher" in (P, O) than all of the other forbidden convex subposets. We begin by introducing some useful notation.

Recall the definition of lexicographic order $\leq_L$ on sequences from page 18. Suppose (P, O) is an oriented poset with $|P| = n$. We define $\mathrm{gr}_{P,O}$, the "greedy" (P, O)-partition, to be the (P, O)-partition σ that maximizes the sequence

$$\left(|\sigma^{-1}(1)|, |\sigma^{-1}(2)|, \ldots, |\sigma^{-1}(n)|\right)$$

in lexicographic order. Therefore, to construct $\mathrm{gr}_{P,O}$, we map as many elements of P as possible to 1, as many elements of the remainder as possible to 2, and so on.

Suppose f is a homogeneous formal power series in the variables $x_1, x_2, \ldots$ of degree n and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a composition of m, where $m \leq n$. We will write

$\lfloor x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \rfloor f$ to mean the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ in f with $x_{k+1}, x_{k+2}, \ldots$ considered as constants. As an example,

$$\lfloor x_1^2 x_2 \rfloor (x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2 + x_1 x_2^2 x_3^2 + x_1^2 x_2 x_5^2) = x_3^2 + x_5^2.$$

If Q is a convex subposet of P, then it will not cause any confusion when we write $O|_Q$ simply as O. In particular, by (Q, O) and $K_{Q,O}(x)$ we mean $(Q, O|_Q)$ and $K_{Q,O|_Q}(x)$ respectively. Finally, we introduce a notion of the dual of an oriented poset (P, O). As usual, let P^* denote the dual poset of P. Define an orientation O^* of P^* by saying that an edge $y \lessdot z$ of P^* is strict in O^* if and only if $z \lessdot y$ is strict in O. This defines a new oriented poset (P^*, O^*).

Lemma 5.6.1. *$K_{P,O}(x)$ is symmetric if and only if $K_{P^*,O^*}(x)$ is symmetric.*

Proof. Suppose $|P| = n$. Since $K_{P,O}(x)$ is quasisymmetric, knowing $K_{P,O}(x)$ is equivalent to knowing $K_{P,O}(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. In fact, $K_{P,O}(x)$ is a symmetric function if and only if $K_{P,O}(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ is a symmetric polynomial in the variables $x_1, x_2, \ldots, x_n$. Hence, for the remainder of this proof, we restrict our attention to (P, O)-partitions and (P^*, O^*)-partitions with images contained in $[n]$. Given such a (P, O)-partition σ, we define a (P^*, O^*)-partition σ^* by $\sigma^*(y) = n + 1 - \sigma(y)$ for all $y \in P$. This gives a bijection between (P, O)-partitions and (P^*, O^*)-partitions. Furthermore

$$\begin{aligned}
K_{P^*,O^*}(x_1, x_2, \ldots, x_n, 0, 0, \ldots) &= \sum_{(P^*,O^*)\text{-partitions } \sigma^*} x_1^{\#\sigma^{*-1}(1)} x_2^{\#\sigma^{*-1}(2)} \cdots x_n^{\#\sigma^{*-1}(n)} \\
&= \sum_{(P,O)\text{-partitions } \sigma} x_1^{\#\sigma^{-1}(n)} x_2^{\#\sigma^{-1}(n-1)} \cdots x_n^{\#\sigma^{-1}(1)} \\
&= \sum_{(P,O)\text{-partitions } \sigma} x_n^{\#\sigma^{-1}(1)} x_{n-1}^{\#\sigma^{-1}(2)} \cdots x_1^{\#\sigma^{-1}(n)} \\
&= K_{P,O}(x_n, x_{n-1}, \ldots, x_1, 0, 0, \ldots)
\end{aligned}$$

Therefore, in the variables $x_1, x_2, \ldots, x_n$, $K_{P^*,O^*}(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ and $K_{P,O}(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ are either both symmetric polynomials or both asymmetric polynomials. We conclude the result. □

Suppose (P, O) is an oriented poset with a forbidden convex subposet B. It is useful to define subposets J_B and I_B of P by

$$J_B = \{y \in P \mid y \geq b \text{ for some } b \in B\}$$

and $I_B = P - J_B$. In particular, the minimal elements of J_B are exactly the minimal elements of B. We are now ready for the results advertised at the beginning of this section.

Proposition 5.6.2. *If the oriented poset (P, O) contains exactly one forbidden convex subposet, then $K_{P,O}(x)$ is not symmetric.*

Proof. Suppose (P, O) contains exactly one forbidden convex subposet B. If $|P| = 3$, then (P, O) must itself be the forbidden convex subposet. As shown in Table 5.2, $K_{P,O}(x)$ is thus not symmetric. Now we let $n = |P|$ and proceed by induction on n.

Case 1: $J_B \neq P$. By our induction hypothesis, $K_{J_B,O}(x)$ is not symmetric. We wish to exploit this fact to show that $K_{P,O}(x)$ is not symmetric. To do this, we consider $\mathrm{gr}_{I_B,O}$, the greedy (I_B, O)-partition, and we suppose that the maximum integer in the image of $\mathrm{gr}_{I_B,O}$ is k. To simplify our expressions, let

$$\left(\mathrm{gr}_{I_B,O}^{-1}(1), \mathrm{gr}_{I_B,O}^{-1}(2), \ldots, \mathrm{gr}_{I_B,O}^{-1}(k)\right) = (\alpha_1, \alpha_2, \ldots, \alpha_k).$$

Now consider $\lfloor x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \rfloor K_{P,O}(x)$, a quasisymmetric function in the variables $x_{k+1}, x_{k+2}, \ldots$. Intuitively, this corresponds to beginning a (P, O)-partition by appropriately mapping an ideal I of size $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ to $[k]$ and we wish to map $J = P - I$ to integers greater than k. We see that

$$\lfloor x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \rfloor K_{P,O}(x) = K_{J_B,O}(x_{k+1}, x_{k+2}, \ldots) + \sum_J a_J K_{J,O}(x_{k+1}, x_{k+2}, \ldots)$$

where the sum is over all subposets $J \neq J_B$ such that $I = P - J$ is an order ideal of P with $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ elements, and where a_J is some non-negative integer. For any such $J \neq J_B$, we have that $j \in I = P - J$ for some $j \in J_B$. In particular, $b \in I$, where b is a minimal element of B and hence of J_B. Therefore, (J, O) contains no forbidden convex subposets, and so by Theorems 5.4.5 and 5.5.1, $K_{J,O}(x)$ is symmetric. However, since $K_{J_B,O}(x)$ is not symmetric, we conclude that $\lfloor x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \rfloor K_{P,O}(x)$ is not symmetric in the variables $x_{k+1}, x_{k+2}, \ldots$. Therefore, $K_{P,O}(x)$ is not symmetric.

Case 2: $J_B = P$. The trick now is to consider the dual oriented poset (P^*, O^*). It contains exactly one forbidden convex subposet, which we will also call B. We define a subposet ${J_B}^*$ of P^* by

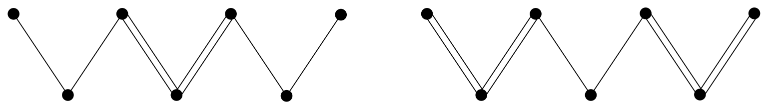
$${J_B}^* = \{y \in P^* \mid y \geq b \text{ in } P^* \text{ for some } b \in B\}$$

If ${J_B}^* \neq P^*$, then we can proceed as above to show that $K_{P^*,O^*}(x)$ is not symmetric. By Lemma 5.6.1, $K_{P,O}(x)$ is then not symmetric, as required. If ${J_B}^* = P^*$, then every element of y of P satisfies $b \leq y \leq b'$ for some elements b and b' of B. Since B is convex, we conclude that $y \in B$ and hence $(P, O) = B$, showing that $K_{P,O}(x)$ is not symmetric. □

The ideas used above can be extended to prove the following result for oriented posets with multiple forbidden convex subposets. Suppose (P, O) is an oriented poset with a forbidden convex subposet B. For convenience, we let

$$\alpha_B = \left(\mathrm{gr}_{I_B,O}^{-1}(1), \mathrm{gr}_{I_B,O}^{-1}(2), \ldots, \mathrm{gr}_{I_B,O}^{-1}(k)\right).$$

Proposition 5.6.3. *Suppose (P, O) be an oriented poset with forbidden convex subposets $B, B_1, B_2, \ldots, B_r$, where $r \geq 0$. If $\alpha_{B_i} <_L \alpha_B$ for $i = 1, 2, \ldots, r$, then $K_{P,O}(x)$ is not symmetric.*



(a) (b)

Figure 5-11: Oriented posets with asymmetric generating functions

Proof. Suppose that $\alpha_B = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. We see that

$$\lfloor x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} \rfloor K_{P,O}(x) = K_{J_B,O}(x_{k+1}, x_{k+2}, \ldots) + \sum_J a_J K_{J,O}(x_{k+1}, x_{k+2}, \ldots)$$

where a_J is some positive integer and where the sum is over all subposets $J \neq J_B$ such that $I = P - J$ is an order ideal of P which has $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ elements and which satisfies $[M_{\alpha_1\alpha_2\cdots\alpha_k}]K_{I,O}(x) \neq 0$. Note that any such I must satisfy

$$(\alpha_1, \alpha_2, \ldots, \alpha_k) \leq_L \left(\mathrm{gr}_{I,O}^{-1}(1), \mathrm{gr}_{I,O}^{-1}(2), \ldots, \mathrm{gr}_{I,O}^{-1}(k)\right).$$

For any such $J \neq J_B$, we have that $j \in I = P - J$ for some $j \in J_B$. In particular, $b \in I$, where b is a minimal element of B and hence of J_B. Furthermore, for any $i = 1, 2, \ldots, r$, since $\alpha_{B_i} <_L (\alpha_1, \alpha_2, \ldots, \alpha_k)$, we get that $b \in I$ for some element b of B_i. For the same reason, $b \in I_B$ for some element b of B_i. Therefore, $J \neq J_B$ contains no forbidden convex subposets and J_B contains B and no other forbidden convex subposets. Hence, in the variables $x_{k+1}, x_{k+2}, \ldots$, $K_{J,O}(x_{k+1}, x_{k+2}, \ldots)$ is symmetric for $J \neq J_B$. However, by Proposition 5.6.2, $K_{J_B,O}(x_{k+1}, x_{k+2}, \ldots)$ is not symmetric. We conclude that $K_{P,O}(x)$ is not symmetric. ☐

Example 5.6.4. Consider the oriented poset (P, O) shown in Figure 5-11(a). We see that it has three forbidden convex subposets, which we denote from left to right by B_1, B_2 and B_3. We have that $\alpha_{B_1} = \alpha_{B_3} = (3, 1)$ while $\alpha_{B_2} = (4)$. Since $\alpha_{B_1} <_L \alpha_{B_2}$, we can apply Proposition 5.6.3. The proof of the proposition implies that $\lfloor x_1^4 \rfloor K_{P,O}(x)$ is an asymmetric function of $x_2, x_3, \ldots$. Indeed, in the variables $x_2, x_3, \ldots$, we can calculate that

$$\lfloor x_1^4 \rfloor K_{P,O}(x) = 2M_3 + 6M_{21} + 7M_{12} + 14M_{111}.$$

Now consider the oriented poset (P, O) shown in Figure 5-11(b). Again it has three forbidden convex subposets, which we denote from left to right by B_1, B_2 and B_3. We have that $\alpha_{B_1} = \alpha_{B_2} = \alpha_{B_3} = (2, 2)$ and so Proposition 5.6.3 does not apply. We will however remark that we do have a general technique which shows that $K_{P,O}(x)$ is not symmetric for this oriented poset. Indeed, let (P, O) denote any oriented poset which has no cycles. By Lemma 5.3.2, (P, O) is a labelled poset. If $|P| = n$, we consider a new labelled poset (P, O^r), whose labels are obtained from those of (P, O) by subtracting each label from $n + 1$. We see that the resulting new orientation O^r is exactly the reverse of the orientation O: strict edges become weak edges and weak edges become strict edges. For example, if (P, O) is the oriented poset of Figure 5-11(b), then (P, O^r) is the oriented poset of Figure 5-11(a). As can be seen from [41, Corollary 7.19.5], at the level of generating functions, $K_{P,O^r}(x) = \omega K_{P,O}(x)$, where ω is the involution on quasisymmetric functions as defined on page 35. Since ω sends symmetric functions to symmetric functions, $K_{P,O^r}(x)$ is symmetric if and only if $K_{P,O}(x)$ is symmetric. Notice that this result is very similar in nature to Lemma 5.6.1.

It is not difficult to extend our approach to prove slightly more general versions of Proposition 5.6.3. As one example, we can combine our greedy labellings of I_B with greedy labellings for $J_B - B$ to give a more refined version of the proposition. However, our current techniques are not sufficient to prove (or disprove) Conjecture 5.2.2 or Conjecture 5.5.4.

Appendix A

Poset terminology

For the following introduction to basic poset terminology, we follow [38, Chapter 3].

A **partially ordered set** (**poset**) P is a set (which by abuse of notation we also call P) together with an order relation $\leq$ satisfying the following properties:

(i) **Reflexivity**: $x \leq x$ for all $x \in P$.

(ii) **Antisymmetry**: If $x \leq y$ and $y \leq x$ for $x, y \in P$, then $x = y$.

(iii) **Transitivity**: If $x \leq y$ and $y \leq z$ for $x, y, z \in P$, then $x \leq z$.

If $x, y \in P$ and there does not exist z satisfying $x < z < y$, then we say that y **covers** x and we write $x \lessdot y$. We will only be concerned with finite posets, and finite posets are completely determined by their cover relations. We will often think of a poset P in terms of its **Hasse diagram**, a graph whose vertices are the elements of P such that if $x < y$ then x is drawn below y, and whose edges are the cover relations. For example, Figure A-1 shows the poset D_{60} of all positive integers that divide 60, where we say $x \leq y$ in D_{60} if y is divisible by x.

The **dual poset** of P is the poset P^* with the same set of elements as P but with the opposite ordering, i.e. $x \leq y$ in P^* if and only if $y \leq x$ in P. The Hasse diagram of P^* can thus be obtained by turning the Hasse diagram of P upside down. Given

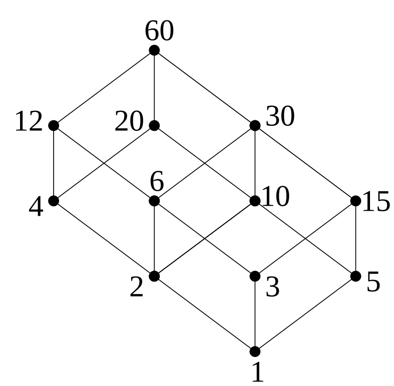


Figure A-1: The Hasse diagram of D_{60}

posets P and Q, a map $\varphi : P \to Q$ is said to be a **poset embedding** if φ is injective and if

$$x \le y \text{ in } P \Leftrightarrow \varphi(x) \le \varphi(y) \text{ in } Q.$$

We then say that φ is a **poset isomorphism** if φ is a bijection.

An **induced subposet** Q of P is a subset Q of P and a partial ordering of Q such that if $x, y \in Q$, then $x \le y$ in Q if and only if $x \le y$ in P. By a **subposet** of P, we will always mean an induced subposet of P. Therefore, to specify a subposet Q of a poset P, it is enough to specify the elements of Q. If $x, y \in P$, with $x \le y$, then we will write $[x, y]$, called an **interval** of P, to denote the subposet of P consisting of all those elements $z \in P$ such that $x \le z \le y$. An **order ideal** (or **down-set**) of a poset P is a subposet I of P such that if $y \in I$ and $x \le y$ in P, then $x \in I$. A subposet of P is said to be a **chain** if any two elements x and y of P are comparable, i.e. either $x \le y$ or $x \ge y$. The **length of a chain** C is defined to be $|C| - 1$. A chain is said to be **maximal** if it is maximal under inclusion. A poset is said to be **graded** if all of its maximal chains have the same length. If P is graded and the length of every maximal chain is n, then we define the **rank function** $\mathrm{rk} : P \to [n]$ of P by $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$ if y covers x in P and $\mathrm{rk}(x) = 0$ if x is a minimal element of P. We define the **rank of a graded poset** to be the rank of any maximal element.

If $x, y \in P$, a **common upper bound** of x and y is an element z of P such that $x \le z$ and $y \le z$. We then say that z is a **least upper bound** if any other common upper bound w of x and y satisfies $w \ge z$. If a least upper bound of x and y exists, then we denote it by $x \vee y$, the **join** of x and y. We will write $z = x \vee_P y$ if we wish to emphasize that z is the join of x and y in the poset P. Similarly, we define the notions of **common lower bound** and **greatest lower bound**. If x and y have a greatest lower bound z in P then we write $z = x \wedge_P y$ or just $z = x \wedge y$, and we say that z is the **meet** of x and y.

A poset P is said to be a **lattice** if every two elements have a meet and a join. A lattice L is said to be **distributive** if, for all $x, y, z \in L$, we have

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

and

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

It is a nice exercise to show that either of these identities implies the other. A **sublattice** of a lattice L is a subposet K of L such that if x and y are in K, then so are $x \vee_L y$ and $x \wedge_L y$. Given a subposet Q of a lattice L, we define the **sublattice generated by** Q to be the smallest sublattice of L which contains Q.

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