POSET EDGE-LABELLINGS AND LEFT MODULARITY

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Abstract. It is known that a graded lattice of rank n is supersolvable if and only if it has an EL-labelling where the labels along any maximal chain are exactly the numbers $1, 2, \ldots, n$ without repetition. These labellings are called S_n EL-labellings, and having such a labelling is also equivalent to possessing a maximal chain of left modular elements. In the case of an ungraded lattice, there is a natural extension of S_n EL-labellings, called interpolating labellings. We show that admitting an interpolating labelling is again equivalent to possessing a maximal chain of left modular elements. Furthermore, we work in the setting of an arbitrary bounded poset as all the above results generalize to this case. We conclude by applying our results to show that the lattice of non-straddling partitions, which is not graded in general, has a maximal chain of left modular elements.

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1. Introduction

An *edge-labelling* of a poset P is a map from the edges of the Hasse diagram of P to $\mathbb{Z}$. Our primary goal is to express certain classical properties of P in terms of edge-labellings admitted by P. The idea of studying edge-labellings of posets goes back to [12]. An important milestone was [3], where A. Björner defined EL-labellings, and showed that if a poset admits an EL-labelling, then it is shellable and hence Cohen-Macaulay. We will be interested in a subclass of EL-labellings, known as S_n EL-labellings. In [13], R. Stanley introduced supersolvable lattices and showed that they admit S_n EL-labellings. Examples of supersolvable lattices include distributive lattices, the lattice of partitions of $[n]$, the lattice of non-crossing partitions of $[n]$ and the lattice of subgroups of a supersolvable group (hence the terminology). It was shown in [9] that a finite graded lattice of rank n is supersolvable if and only if it admits an S_n EL-labelling. In many ways, this characterization of lattice supersolvability in terms of edge-labellings serves as the starting point for our investigations.

For basic definitions concerning partially ordered sets, see [14]. We will say that a poset P is *bounded* if it contains a unique minimal element and a unique maximal element, denoted $\hat{0}$ and $\hat{1}$ respectively. All the posets we will consider will be finite and bounded. A chain of a poset P is said to be *maximal* if it is maximal under inclusion. We say that P is *graded* if all the maximal chains of P have the same length, and we call this length the *rank* of P. We will write $x \lessdot y$ if y covers x in P and $x \leq y$ if y either covers or equals x. The edge-labelling γ of P is said to be an *EL-labelling* if for any $y < z$ in P,

(i) there is a unique unrefinable chain $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$ such that $\gamma(w_0, w_1) \leq \gamma(w_1, w_2) \leq \cdots \leq \gamma(w_{r-1}, w_r)$, and

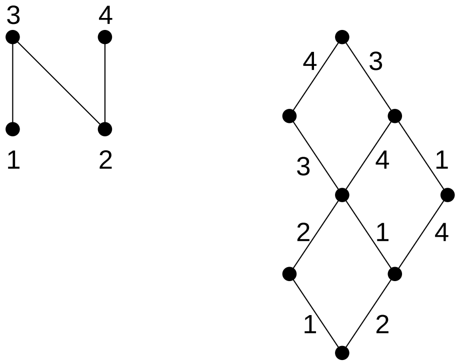


Figure 1

(ii) the sequence of labels of this chain (referred to as the *increasing chain* from y to z), when read from bottom to top, lexicographically precedes the labels of any other unrefinable chain from y to z.

This concept originates in [3]; for the case where P is not graded, see [4, 5]. If P is graded of rank n with an EL-labelling γ, then γ is said to be an S_n *EL-labelling* if the labels along any maximal chain of P are all distinct and are elements of $[n]$. In other words, for every maximal chain $\hat{0} = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_n = \hat{1}$ of P, the map sending i to $\gamma(w_{i-1}, w_i)$ is a permutation of $[n]$. Note that the second condition in the definition of an EL-labelling is redundant in this case.

Example 1.1. Any finite distributive lattice has an S_n EL-labelling. Let L be a finite distributive lattice of rank n. By the Fundamental Theorem of Finite Distributive Lattices [2, p. 59, Thm. 3], that is equivalent to saying that $L = J(Q)$, the lattice of order ideals of some n-element poset Q. Let $\omega : Q \to [n]$ be a linear extension of Q, i.e., any bijection labelling the vertices of Q that is order-preserving (if $a < b$ in Q then $\omega(a) < \omega(b)$). This labelling of the vertices of Q defines a labelling of the edges of $J(Q)$ as follows. If y covers x in $J(Q)$, then the order ideal corresponding to y is obtained from the order ideal corresponding to x by adding a single element, labeled by i, say. Then we set $\gamma(x, y) = i$. This gives us an S_n EL-labelling for $L = J(Q)$. Figure 1 shows a labelled poset and its lattice of order ideals with the appropriate edge-labelling.

A finite lattice L is said to be *supersolvable* if it contains a maximal chain, called an *M-chain* of L, which together with any other chain in L generates a distributive sublattice. We can label each such distributive sublattice by the method described in Example 1.1 in such a way that the M-chain is the unique increasing maximal chain. As shown in [13], this will assign a unique label to each edge of L and the resulting global labelling of L is an S_n EL-labelling.

There is also a characterization of lattice supersolvability in terms of left modularity. Given an element x of a finite lattice L, and a pair of elements $y \leq z$, it is always true that

$$(x \vee y) \wedge z \geq (x \wedge z) \vee y. \qquad (1)$$

The element x is said to be *left modular* if, for all $y \leq z$, equality holds in (1). Following A. Blass and B. Sagan [6], we will say that a lattice itself is *left modular* if it contains a left modular maximal chain, that is, a maximal chain each of whose elements is left modular. (One might guess that we should define a lattice to be left modular if all of its elements are left modular, but this is equivalent to the definition of a modular lattice.) As shown in [13], any M-chain of a supersolvable lattice is always a left modular maximal chain, and so supersolvable lattices are left modular. Furthermore, it is shown by L. S.-C. Liu [7] that if L is a finite graded lattice with a left modular maximal chain M, then L has an S_n EL-labelling with increasing maximal chain M. In turn, as shown in [9], this implies that L is supersolvable, and so we conclude the following.

Theorem 1. *Let L be a finite graded lattice of rank n. Then the following are equivalent:*

(1) *L has an S_n EL-labelling,*
(2) *L is left modular,*
(3) *L is supersolvable.*

It is shown in [13] that if L is upper-semimodular, then L is left modular if and only if L is supersolvable. Theorem 1 is a considerable strengthening of this. Here we used S_n EL-labellings to connect left modularity and supersolvability. It is natural to ask for a more direct proof that (2) implies (3); such a proof has recently been provided by the second author in [15].

Our goal is to generalize Theorem 1 to the case when L is not graded and, moreover, to the case when L is not necessarily a lattice. We now wish to define natural generalizations of S_n EL-labellings and of maximal left modular chains.

Definition 1.2. An EL-labelling γ of a poset P is said to be *interpolating* if, for any $y \lessdot u \lessdot z$, either

(i) $\gamma(y,u) < \gamma(u,z)$ or
(ii) the increasing chain from y to z, say $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$, has the properties that its labels are strictly increasing and that $\gamma(w_0,w_1) = \gamma(u,z)$ and $\gamma(w_{r-1},w_r) = \gamma(y,u)$.

Example 1.3. The reader is invited to check that the labelling of the non-graded poset shown in Figure 2 is an interpolating EL-labelling. In fact, the poset shown is the so-called "Tamari lattice" T_4. For all positive integers n, there exists a Tamari lattice T_n with C_n elements, where $C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number. More information on the Tamari lattice can be found in [5, §9], [6, §7] and the references given there, and in [7, §3.2], where this interpolating EL-labelling appears. The Tamari lattice is shown to have an EL-labelling in [5] and is shown to be left modular in [6].

If P is graded of rank n and has an interpolating labelling γ in which the labels on the increasing maximal chain reading from bottom to top are $1, 2, \ldots n$, then we can check (cf. Lemma 3.2) that γ is an S_n EL-labelling.

Our next step is to define left modularity in the non-lattice case. Let x and y be elements of P. We know that x and y have at least one common upper bound, namely $\hat{1}$. If the set of common upper bounds of x and y has a least element, then we denote it by $x \vee y$. Similarly, if x and y have a greatest common lower bound, then we denote it by $x \wedge y$.

Now let w and z be elements of P with $w, z \geq y$. Consider the set of common lower bounds for w and z that are also greater than or equal to y. Clearly, y is in this set. If this set has a greatest element, then we denote it by $w \wedge_y z$ and we say

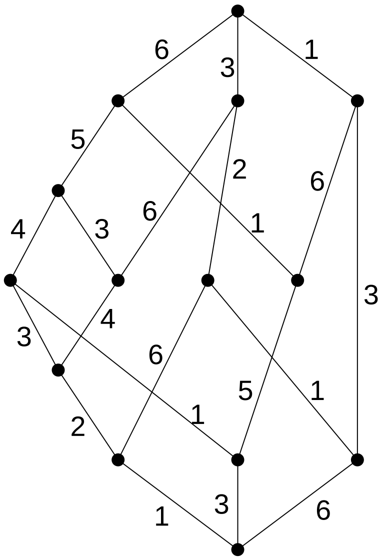


FIGURE 2. The Tamari lattice T_4 and its interpolating EL-labelling

that $w \wedge_y z$ is well-defined (in $[y, \hat{1}]$). We see that $(x \vee y) \wedge_y z$ is well-defined in the poset shown in Figure 3, even though $(x \vee y) \wedge z$ is not. Similarly, let w and y be elements of P with $w, y \leq z$. If the set $\{u \in P \mid u \geq w, y \text{ and } u \leq z\}$ has a least element, then we denote it by $w \vee^z y$ and we say that $w \vee^z y$ is well-defined (in $[\hat{0}, z]$). We will usually be interested in expressions of the form $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$. The reader that is solely interested in the lattice case can choose to ignore the subscripts and superscripts on the meet and join symbols.

Definition 1.4. An element x of a poset P is said to be *viable* if, for all $y \leq z$ in P, $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ are well-defined. A maximal chain of P is said to be viable if each of its elements is viable.

Example 1.5. The poset shown in Figure 3 is certainly not a lattice but the reader can check that the increasing maximal chain is viable.

Definition 1.6. A viable element x of a poset P is said to be *left modular* if, for all $y \leq z$ in P,

$$(x \vee y) \wedge_y z = (x \wedge z) \vee^z y.$$

A maximal chain of P is said to be left modular if each of its elements is viable and left modular, and P is said to be left modular if it possesses a left modular maximal chain.

This brings us to the first of our main theorems.

Theorem 2. *Let P be a bounded poset with a left modular maximal chain M. Then P has an interpolating EL-labelling with M as its increasing maximal chain.*

The proof of this theorem will be the content of the next section. In Section 3, we will prove the following converse result.

Theorem 3. *Let P be a bounded poset with an interpolating EL-labelling. The unique increasing chain from $\hat{0}$ to $\hat{1}$ is a left modular maximal chain.*

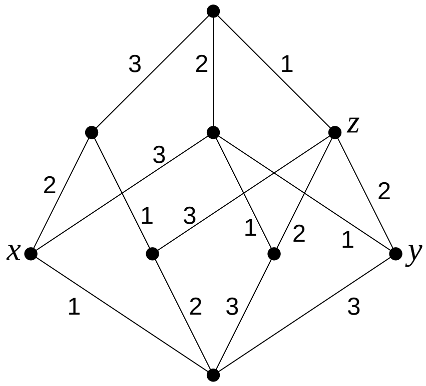


Figure 3

These two theorems, when compared with Theorem 1, might lead one to ask about possible supersolvability results for bounded posets that aren't graded lattices. This problem is discussed in Section 4. In the case of graded posets, we obtain a satisfactory result, namely Theorem 4. As a consequence, we have given an answer to the question of when a graded poset P has an S_n EL-labelling. This has ramifications on the existence of a "good 0-Hecke algebra action" on the maximal chains of the poset, as discussed in [9]. However, it remains an open problem to appropriately extend the definition of supersolvability to ungraded posets.

An explicit application of Theorem 3 is the subject of Section 5. As a variation on non-crossing partitions and non-nesting partitions, we define non-straddling partitions. Ordering the set of non-straddling partitions of $[n]$ by refinement gives a poset, denoted NS_n, that is generally a non-graded lattice. We define an edge-labelling γ for NS_n that is analogous to the usual EL-labelling for the lattice of partitions of $[n]$. In order to show that NS_n is left modular, we then prove that γ is an interpolating EL-labelling.

2. Proof of Theorem 2

Throughout this section, we suppose that P is a bounded poset with a left modular maximal chain $M : \hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$. We want to show that P has an interpolating EL-labelling. Our approach will be as follows: we will begin by specifying an edge-labelling γ for P such that M is an increasing chain with respect to γ. We will then prove a series of lemmas which build on the viability and left modularity properties. These culminate with Proposition 2.6 which, roughly speaking, gives a more local definition for γ. We will then be ready to show that γ is an EL-labelling and is, furthermore, an interpolating EL-labelling.

We choose a label set $l_1 < \cdots < l_n$ of natural numbers. (For most purposes, we can let $l_i = i$.) We define an edge-labelling γ on P by setting $\gamma(y, z) = l_i$ for $y \lessdot z$ if

$$(x_{i-1} \vee y) \wedge_y z = y \quad \text{and} \quad (x_i \vee y) \wedge_y z = z.$$

It is easy to see that γ is well-defined. We will refer to it as the labelling induced by M and the label set $\{l_i\}$. When P is a lattice, this labelling appears, for example, in [7, 16]. As in [7], we can give an equivalent definition of γ as follows.

Lemma 2.1. *Suppose $y \lessdot z$ in P. Then $\gamma(y,z) = l_i$ if and only if*

$$i = \min\{j \mid x_j \vee y \geq z\} = \max\{j+1 \mid x_j \wedge z \leq y\}.$$

Proof. That $i = \min\{j \mid x_j \vee y \geq z\}$ is immediate from the definition of γ. By left modularity, $\gamma(y,z) = l_i$ if and only if $(x_{i-1} \wedge z) \vee^z y = y$ and $(x_i \wedge z) \vee^z y = z$. In other words, $x_{i-1} \wedge z \leq y$ and $x_i \wedge z \not\leq y$. It follows that $i = \max\{j+1 \mid x_j \wedge z \leq y\}$. □

Lemma 2.2. *Suppose that $y \leq w \leq z$ in P and let $x \in M$. Then $((x \wedge z) \vee^z y) \vee^z w$ is well-defined and equals $(x \wedge z) \vee^z w$. Similarly, $((x \vee y) \wedge_y z) \wedge_y w$ is well-defined and equals $(x \vee y) \wedge_y w$.*

Proof. It is routine to check that, in $[\hat{0}, z]$, $(x \wedge z) \vee^z w$ is the least common upper bound for w and $(x \wedge z) \vee^z y$, and that, in $[y, \hat{1}]$, $(x \vee y) \wedge_y w$ is the greatest common lower bound lower bound for $(x \vee y) \wedge_y z$ and w. □

Lemma 2.3. *Suppose that $t \leq u$ in $[y,z]$ and $x \in M$. Let $w = (x \vee y) \wedge_y z = (x \wedge z) \vee^z y$ in $[y,z]$. Then $(w \vee^z t) \wedge_t u$ and $(w \wedge_y u) \vee^u t$ are well-defined elements of $[t,u]$ and are equal.*

Proof. We see that, by Lemma 2.2,

$$\begin{aligned}(x \vee t) \wedge_t u &= ((x \vee t) \wedge_t z) \wedge_t u = ((x \wedge z) \vee^z t) \wedge_t u \\ &= (((x \wedge z) \vee^z y) \vee^z t) \wedge_t u = (w \vee^z t) \wedge_t u.\end{aligned}$$

Similarly,

$$(x \wedge u) \vee^u t = (w \wedge_y u) \vee^u t$$

But $(x \vee t) \wedge_t u = (x \wedge u) \vee^u t$, yielding the result. □

Lemma 2.4. *Suppose x and w are viable and that x is left modular in P.*

(a) *If $x \lessdot w$ then for any z in P we have $x \wedge z \leq w \wedge z$.*
(b) *If $w \lessdot x$ then for any y in P we have $w \vee y \leq x \vee y$.*

Part (b) appears in the lattice case in [7, Lemma 2.5.6] and [8, Lemma 5.3].

Proof. We prove (a); (b) is similar. Assume, seeking a contradiction, that $x \wedge z < u < w \wedge z$ for some $u \in P$. Now $u \leq z$ and $u \leq w$. It follows that $u \not\leq x$.

Now $x < x \vee u \leq w$. Therefore, $w = x \vee u$. So

$$u = (x \wedge z) \vee^z u = (x \vee u) \wedge_u z = w \wedge z,$$

which is a contradiction. □

We now prove a slight extension of [7, Lemma 2.5.7] and [8, Lemma 5.4].

Lemma 2.5. *The elements of $[y,z]$ of the form $(x_i \vee y) \wedge_y z$ form a left modular maximal chain in $[y,z]$.*

Proof. Lemma 2.3 gives the viability and left modularity properties. By Lemma 2.4(b), $x_i \vee y \leq x_{i+1} \vee y$. By Lemma 2.3 with $z = \hat{1}$, we have that $x_i \vee y$ is left modular in $[y, \hat{1}]$. Therefore, $(x_i \vee y) \wedge_y z \leq (x_{i+1} \vee y) \wedge_y z$ by Lemma 2.4(a). □

We are now ready for the last, and most important, of our preliminary results. Let $[y,z]$ be an interval in P. We call the maximal chain of $[y,z]$ from Lemma 2.5 the *induced* left modular maximal chain of $[y,z]$. One way to get a second edge-labelling for $[y,z]$ would be to take the labelling induced in $[y,z]$ by this induced maximal chain. We now prove that, for a suitable choice of label set, this labelling coincides with γ.

Proposition 2.6. *Let P be a bounded poset, $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ a left modular maximal chain and γ the corresponding edge-labelling with label set $\{l_i\}$. Let $y < z$, and define c_i by saying*

$$\begin{aligned} y &= (x_0 \vee y) \wedge_y z = \cdots = (x_{c_1 - 1} \vee y) \wedge_y z \\ &\lessdot (x_{c_1} \vee y) \wedge_y z = \cdots = (x_{c_2 - 1} \vee y) \wedge_y z \lessdot \cdots \\ &\lessdot (x_{c_r} \vee y) \wedge_y z = \cdots = (x_n \vee y) \wedge_y z. \end{aligned}$$

Let $m_i = l_{c_i}$. Let δ be the labelling of $[y, z]$ induced by its induced left modular maximal chain and the label set $\{m_i\}$. Then δ agrees with γ restricted to the edges of $[y, z]$.

Proof. Suppose $t \lessdot u$ in $[y, z]$. Using ideas from the proof of Lemma 2.3,

$$\begin{aligned} \delta(t, u) = m_i \quad &\Leftrightarrow \quad (((x_{c_i - 1} \vee y) \wedge_y z) \vee^z t) \wedge_t u = t \text{ and} \\ &\qquad\ (((x_{c_i} \vee y) \wedge_y z) \vee^z t) \wedge_t u = u \\ &\Leftrightarrow \quad (x_{c_i - 1} \vee t) \wedge_t u = t \text{ and } (x_{c_i} \vee t) \wedge_t u = u \\ &\Leftrightarrow \quad \gamma(t, u) = l_{c_i}. \end{aligned}$$

□

Proof of Theorem 2. We now know that the induced left modular chain in $[y, z]$ has (strictly) increasing labels, say $m_1 < m_2 < \cdots < m_r$. Our first step is to show that it is the only maximal chain with (weakly) increasing labels. Suppose that $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$ is the induced chain and that $y = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_s = z$ is another chain with increasing labels.

If $s = 1$ then $y \lessdot z$ and the result is clear. Suppose $s \geq 2$. By Proposition 2.6, we may assume that the labelling on $[y, z]$ is induced by the induced left modular chain $\{w_i\}$. In particular, we have that $\gamma(u_i, u_{i+1}) = m_l$ where $l = \min\{j \mid w_j \vee^z u_i \geq u_{i+1}\}$. Let k be the least number such that $u_k \geq w_1$. Then it is clear that $\gamma(u_{k-1}, u_k) = m_1$. Note that this is the smallest label that can occur on any edge in $[y, z]$. Since the labels on the chain $\{u_i\}$ are assumed to be increasing, we must have $\gamma(u_0, u_1) = m_1$. It follows that $w_1 \vee^z u_0 \geq u_1$ and since $y \lessdot w_1$, we must have $u_1 = w_1$. Thus, by induction, the two chains coincide. We conclude that the induced left modular maximal chain is the only chain in $[y, z]$ with increasing labels.

It also has the lexicographically least set of labels. To see this, suppose that $y = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_s = z$ is another chain in $[y, z]$. We assume that $u_1 \neq w_1$ since, otherwise, we can just restrict our attention to $[u_1, z]$. We have $\gamma(u_0, u_1) = m_l$, where $l = \min\{j \mid w_j \geq u_1\} \geq 2$ since $w_1 \not\geq u_1$. Hence $\gamma(u_0, u_1) \geq m_2 > \gamma(w_0, w_1)$. This gives that γ is an EL-labelling. (That γ is an EL-labelling was already shown in the lattice case in [7, 16].)

Finally, we show that it is an interpolating EL-labelling. If $y \lessdot u \lessdot z$ is not the induced left modular maximal chain in $[y, z]$, then let $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$ be the induced left modular maximal chain. We have that $\gamma(y, u) = m_l$ where

$$l = \min\{j \mid w_j \vee^z y \geq u\} = \min\{j \mid w_j \geq u\} = r$$

since $u \lessdot z$. Therefore, $\gamma(y, u) = m_r$. Also, $\gamma(u, z) = m_l$ where

$$l = \max\{j + 1 \mid w_j \wedge_y z \leq u\} = \max\{j + 1 \mid w_j \leq u\} = 1$$

since $y \lessdot u$. Therefore, $\gamma(y, u) = m_1$, as required. □

3. Proof of Theorem 3

We suppose that P is a bounded poset with an interpolating EL-labelling γ. Let $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ be the increasing chain from $\hat{0}$ to $\hat{1}$ and let $l_i = \gamma(x_{i-1}, x_i)$. We will begin by establishing some basic facts about interpolating labellings. These results will enable us to show certain meets and joins exist by looking at the labels that appear along particular increasing chains. We will thus show that the x_i are viable. We will finish by showing that the x_i are left modular, again by looking at the labels on increasing chains.

Let $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$. Suppose that, for some i, we have $\gamma(w_{i-1}, w_i) > \gamma(w_i, w_{i+1})$. Then the "basic replacement" at i takes the given chain and replaces the subchain $w_{i-1} \lessdot w_i \lessdot w_{i+1}$ by the increasing chain from w_{i-1} to w_{i+1}. The basic tool for dealing with interpolating labellings is the following well-known fact about EL-labellings.

Lemma 3.1. *Let $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$. Successively perform basic replacements on this chain, and stop when no more basic replacements can be made. This algorithm terminates, and yields the increasing chain from y to z.*

Proof. At each step, the sequence of labels on the new chain lexicographically precedes the sequence on the old chain, so the process must terminate, and it is clear that it terminates in an increasing chain. □

We now prove some simple consequences of this lemma.

Lemma 3.2. *Let m be the chain $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$. Then the labels on m all occur on the increasing chain from y to z and are all different. Furthermore, all the labels on the increasing chain from y to z are bounded between the lowest and highest labels on m.*

Proof. That the labels on the given chain all occur on the increasing chain follows immediately from Lemma 3.1 and the fact that after a basic replacement, the labels on the old chain all occur on the new chain. Similar reasoning implies that the labels on the increasing chain are bounded between the lowest and highest labels on m.

That the labels are all different again follows from Lemma 3.1. Suppose otherwise. By repeated basic replacements, one obtains a chain which has two successive equal labels, which is not permitted by the definition of an interpolating labelling. □

Lemma 3.3. *Let $z \in P$ such that there is some chain from $\hat{0}$ to z all of whose labels are in $\{l_1, \ldots, l_i\}$. Then $z \leq x_i$. Conversely, if $z \leq x_i$, then all the labels on any chain from $\hat{0}$ to z are in $\{l_1, \ldots, l_i\}$.*

Proof. We begin by proving the first statement. By Lemma 3.2, the labels on the increasing chain from $\hat{0}$ to z are in $\{l_1, \ldots, l_i\}$. Find the increasing chain from z to $\hat{1}$. Let w be the element in that chain such that all the labels below it on the chain are in $\{l_1, \ldots, l_i\}$, and those above it are in $\{l_{i+1}, \ldots, l_n\}$. Again, by Lemma 3.2, the increasing chain from $\hat{0}$ to w has all its labels in $\{l_1, \ldots, l_i\}$, and the increasing chain from w to $\hat{1}$ has all its labels in $\{l_{i+1}, \ldots, l_n\}$. Thus w is on the increasing chain from $\hat{0}$ to $\hat{1}$, and so $w = x_i$. But by construction $w \geq z$. So $x_i \geq z$.

To prove the converse, observe that by Lemma 3.2, no label can occur more than once on any chain. But since every label in $\{l_{i+1}, \ldots, l_n\}$ occurs on the increasing chain from x_i to $\hat{1}$, no label from among that set can occur on any edge below x_i. ☐

The obvious dual of Lemma 3.3 is proved similarly:

Corollary 3.4. *Let $z \in P$ such that there is some chain from z to $\hat{1}$ all of whose labels are in $\{l_{i+1}, \ldots, l_n\}$. Then $z \geq x_i$. Conversely, if $z \geq x_i$, then all the labels on any chain from z to $\hat{1}$ are in $\{l_{i+1}, \ldots, l_n\}$.*

We are now ready to prove the necessary viability properties.

Lemma 3.5. *$x_i \vee z$ and $x_i \wedge z$ are well-defined for any $z \in P$ and for $i = 1, 2, \ldots, n$.*

Proof. We will prove that $x_i \wedge z$ is well-defined. The proof that $x_i \vee z$ is well-defined is similar. Let w be the maximum element on the increasing chain from $\hat{0}$ to z such that all labels on the increasing chain between $\hat{0}$ and w are in $\{l_1, \ldots, l_i\}$. Clearly $w \leq z$ and, by Lemma 3.3, $w \leq x_i$.

Suppose $y \leq z, x_i$. It follows that all labels from $\hat{0}$ to y are in $\{l_1, \ldots, l_i\}$. Consider the increasing chain from y to z. There exists an element u on this chain such that all the labels on the increasing chain from $\hat{0}$ to u are in $\{l_1, \ldots, l_i\}$ and all the labels on the increasing chain from u to z are in $\{l_{i+1}, \ldots, l_n\}$. Therefore, u is on the increasing chain from $\hat{0}$ to z and, in fact, $u = w$. Also, we have that $\hat{0} \leq y \leq u = w \leq z$. We conclude that w is the greatest common lower bound for z and x_i. ☐

Lemma 3.6. *$\hat{0} = x_0 \wedge z \leq x_1 \wedge z \leq \cdots \leq x_n \wedge z = z$, after we delete repeated elements, is the increasing chain in $[\hat{0}, z]$. Hence, $(x_i \wedge z) \vee^z y$ is well-defined for $y \leq z$. Similarly, $(x_i \vee y) \wedge_y z$ is well-defined.*

Proof. From the previous proof, we know that $x_i \wedge z$ is the maximum element on the increasing chain from $\hat{0}$ to z such that all labels on the increasing chain between $\hat{0}$ and $x_i \wedge z$ are in $\{l_1, \ldots, l_i\}$. The first assertion follows easily from this.

Now apply Lemma 3.5 to the bounded poset $[\hat{0}, z]$. It has an obvious interpolating labelling induced from the interpolating labelling of P. Recall that our definition of the existence of $(x_i \wedge z) \vee^z y$ only requires it to be well-defined in $[\hat{0}, z]$. The result follows. ☐

We conclude that the increasing maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ of P is viable. It remains to show that it is left modular.

Proof of Theorem 3. Suppose that x_i is not left modular for some i. Then there exists some pair $y \leq z$ such that $(x_i \vee y) \wedge_y z > (x_i \wedge z) \vee^z y$. Set $x = x_i$, $b = (x_i \wedge z) \vee^z y$ and $c = (x_i \vee y) \wedge_y z$. Observe that $d := x \vee b \geq c$ while $a := x \wedge c \leq b$. So the picture is as shown in Figure 4.

By Lemma 3.3, the labels on the increasing chain from $\hat{0}$ to a are less than or equal to l_i. Consider the increasing chain from a to c. Let w be the first element along the chain. If $\gamma(a, w) \leq l_i$, then by Lemma 3.3, $w \leq x_i$, contradicting the fact that $a = x \wedge c$. Thus the labels on the increasing chain from a to c are all greater than l_i. Dually, the labels on the increasing chain from b to d are less than or equal to l_i. But now, by Lemma 3.2, the labels on the increasing chain from b to c must be contained in the labels on the increasing chain from a to c, and also from b to d. But there are no such labels, implying a contradiction. We conclude that the x_i are all left modular. ☐

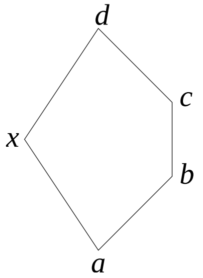


FIGURE 4

We have shown that if P is a bounded poset with an interpolating labelling γ, then the unique increasing maximal chain M is a left modular maximal chain. By Theorem 2, M then induces an interpolating EL-labelling of P. We now show that this labelling agrees with γ for a suitable choice of label set, which is a special case of the following proposition.

Proposition 3.7. *Let γ and δ be two interpolating EL-labellings of a bounded poset P. If γ and δ agree on the γ-increasing chain from $\hat{0}$ to $\hat{1}$, then γ and δ coincide.*

Proof. Let $m : \hat{0} = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = \hat{1}$ be the maximal chain with the lexicographically first γ labelling among those chains for which γ and δ disagree. Since m is not the γ-increasing chain from $\hat{0}$ to $\hat{1}$, we can find an i such that $\gamma(w_{i-1}, w_i) > \gamma(w_i, w_{i+1})$. Let m' be the result of the basic replacement at i with respect to the labelling γ. Then the γ-label sequence of m' lexicographically precedes that of m, so γ and δ agree on m'. But using the fact that γ and δ are interpolating, it follows that they also agree on m. Thus they agree everywhere. ☐

4. GENERALIZING SUPERSOLVABILITY

Suppose P is a bounded poset. For now, we consider the case of P being graded of rank n. We would like to define what it means for P to be *supersolvable*, thus generalizing Stanley's definition of lattice supersolvability. A definition of poset supersolvability with a different purpose appears in [16] but we would like a more general definition. In particular, we would like P to be supersolvable if and only if P has an S_n EL-labelling. For example, the poset shown in Figure 3, while it doesn't satisfy V. Welker's definition, should satisfy our definition. We need to define, in the poset case, the equivalent of a sublattice generated by two chains.

Suppose P has a viable maximal chain M. Thus $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ are well-defined for $x \in M$ and $y \leq z$ in P. Given any chain c of P, we define $R_M(c)$ to be the smallest subposet of P satisfying the following two conditions:

(i) M and c are contained in $R_M(c)$,
(ii) If $y \leq z$ in P and y and z are in $R_M(c)$, then so are $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ for any x in M.

Definition 4.1. We say that a bounded poset P is *supersolvable* with M-chain M if M is a viable maximal chain and $R_M(c)$ is a distributive lattice for any chain c of P.

Since distributive lattices are graded, it is clear that a poset must be graded in order to be supersolvable. We now come to the main result of this section.

Theorem 4. *Let P be a bounded graded poset of rank n. Then the following are equivalent:*

1. *P has an S_n EL-labelling,*
2. *P is left modular,*
3. *P is supersolvable.*

Proof. Observe that for a graded poset, Lemma 3.2 implies that an interpolating labelling is an S_n EL-labelling, and the converse is obvious. Thus, Theorems 2 and 3 restricted to the graded case give us that (1) $\Leftrightarrow$ (2).

Our next step is to show that (1) and (2) together imply (3). Suppose P is a bounded graded poset of rank n with an S_n EL-labelling. Let M denote the increasing maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ of P. We also know that M is viable and left modular and induces the same S_n EL-labelling. Given any maximal chain m of P, we define $Q_M(m)$ to be the closure of m in P under basic replacements. In other words, $Q_M(m)$ is the smallest subposet of P which contains M and m and which has the property that, if y and z are in $Q_M(m)$ with $y \leq z$, then the increasing chain between y and z is also in $Q_M(m)$. It is shown in [9, Proof of Thm. 1] that $Q_M(m)$ is a distributive lattice. There P is a lattice but the proof of distributivity doesn't use this fact. Now consider $R_M(c)$. We will show that there exists a maximal chain m of P such that $R_M(c) = Q_M(m)$. Let m be the maximal chain of P which contains c and which has increasing labels between successive elements of $c \cup \{\hat{0}, \hat{1}\}$. The only idea we need is that, for $y \leq z$ in P, the increasing chain from y to z is given by $y = (x_0 \vee y) \wedge_y z \leq (x_1 \vee y) \wedge_y z \leq \cdots \leq (x_n \vee y) \wedge_y z = z$, where we delete repeated elements. This follows from Lemma 2.5 since the induced left modular chain in $[y, z]$ has increasing labels. It now follows that $R_M(c) = Q_M(m)$, and hence $R_M(c)$ is a distributive lattice.

Finally, we will show that (3) $\Rightarrow$ (2). We suppose that P is a bounded supersolvable poset with M-chain M. Suppose $y \leq z$ in P and let c be the chain $y \leq z$. For any x in M, $x \vee y$ is well-defined in P (because M is assumed to be viable) and equals the usual join of x and y in the lattice $R_M(c)$. The same idea applies to $x \wedge z$, $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$. Since $R_M(c)$ is distributive, we have that

$$(x \vee y) \wedge_y z = (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) = (x \wedge z) \vee y = (x \wedge z) \vee^z y$$

in $R_M(c)$ and so M is left modular in P. $\square$

Remark 4.2. We know from Theorem 1 that a graded lattice of rank n is supersolvable if and only if it has an S_n EL-labelling. Therefore, it follows from Theorem 4 that the definition of a supersolvable poset when restricted to graded lattices yields the usual definition of a supersolvable lattice. (Note that this is not *a priori* obvious from our definition of a supersolvable poset.)

Remark 4.3. The argument above for the equality of $R_M(c)$ and $Q_M(m)$ holds even if P is not graded. However, in the ungraded case, it is certainly not true that $Q_M(m)$ is distributive. The search for a full generalization of Theorem 1 thus leads us to ask what can be said about $Q_M(m)$ in the ungraded case. Is it a lattice? Can we say anything even in the case that P is a lattice?

5. NON-STRADDLING PARTITIONS

Let Π_n denote the lattice of partitions of the set $[n]$ into blocks, where we order partitions by refinement: if y and z are partitions of $[n]$ we say that $y \leq z$ if every block of y is contained in some block of z. Equivalently, z covers y in Π_n if z is obtained from y by merging two blocks of y. Therefore, Π_n is graded of rank $n-1$. Π_n is shown to be supersolvable in [13] and hence has an S_{n-1} EL-labelling, which we denote be δ. In fact, it will simplify our discussion if we use the label set $\{2,\ldots,n\}$ for δ, rather than the label set $[n-1]$. We choose the M-chain, and hence the increasing maximal chain for δ, to be the maximal chain consisting of the bottom element and those partitions of $[n]$ whose only non-singleton block is $[i]$, where $2 \leq i \leq n$. In the literature, δ is often defined in the following form, which can be shown to be equivalent. If z is obtained from y by merging the blocks B and B', then we set

$$\delta(y,z) = \max\{\min B, \min B'\}.$$

For any $x \in \Pi_n$, we will say that $j \in \{2,\ldots n\}$ is a *block minimum* in x if $j = \min B$ for some block B of x. In particular, we see that $\delta(y,z)$ is the unique block minimum in y that is not a block minimum in z.

Recall that a *non-crossing* partition of $[n]$ is a partition with the property that if some block B contains a and c and some block B' contains b and d with $a < b < c < d$, then $B = B'$. Again, we can order the set of non-crossing partitions of $[n]$ by refinement and we denote the resulting poset by NC_n. This poset, which can be shown to be a lattice, has many nice properties and has been studied extensively. More information can be found in R. Simion's survey article [11] and the references given there. Since NC_n is a subposet of Π_n, we can consider δ restricted to the edges of NC_n. It was observed by Björner and P. Edelman in [3] that this gives an EL-labelling for NC_n and we can easily see that this EL-labelling is, in fact, an S_{n-1} EL-labelling (once we subtract 1 from every label).

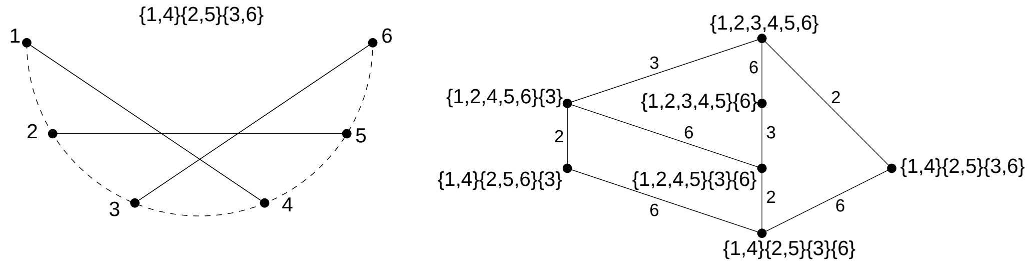
We are now ready to state our main definition for this section, which should be compared with the definition above of non-crossing partitions.

Definition 5.1. A partition of $[n]$ is said to be *non-straddling* if whenever some block B contains a and d and some block B' contains b and c with $a < b < c < d$, then $B = B'$.

This definition is also very similar to that of *non-nesting* partitions, as defined by A. Postnikov and discussed in [10, Remark 2] and [1]. The only difference in the definition of non-nesting partitions is that we do not require $B = B'$ if there is also an element of B between b and c. So, for example, $\{1,3,5\}\{2,4\}$ is a non-nesting partition in Π_5 but is not a non-straddling partition. We say that $\{1,3,5\}\{2,4\}$ is a *straddling partition*, that $1 < 2 < 4 < 5$ is a *straddle*, and that the blocks $\{1,3,5\}$ and $\{2,4\}$ form a straddle.

Let NS_n be the subposet of Π_n consisting of those partitions that are non-straddling. To distinguish the interval $[x,y]$ in Π_n from the interval $[x,y]$ in NS_n, we will use the notation $[x,y]_{\Pi_n}$ and $[x,y]_{NS_n}$, respectively. We note that the meet in Π_n of two non-straddling partitions is again non-straddling, implying that NS_n is a meet-semilattice. Since $\{1,2.\ldots,n\}$ is a top element for NS_n, we conclude that NS_n is a lattice. On the other hand, NS_n is not graded. For example, consider those elements of Π_6 that cover $\{1,4\}\{2,5\}\{3,6\}$, as represented in Figure 5(a). $\{1,2,4,5\}\{3,6\}$, $\{1,3,4,6\}\{2,5\}$ and $\{1,4\}\{2,3,5,6\}$ are all straddling partitions, so $\{1,4\}\{2,5\}\{3,6\}$ is covered in NS_6 by $\{1,2,3,4,5,6\}$. Figure 5(b) shows $[\{1,4\}\{2,5\}\{3\}\{6\},\hat{1}]_{NS_6}$.

Therefore, unlike Π_n and NC_n, NS_n cannot have an S_{n-1} EL-labelling. However, we can ask if it has an interpolating EL-labelling. We see that the following



(a) (b)

Figure 5

three ways of defining an edge-labelling γ for NS_n are equivalent. Observe that if $y \lessdot z$ in NS_n, then z is obtained from y by merging the blocks $B_1, B_2, \ldots, B_r$ of y into a single block B in z. We set

$$\begin{aligned} \gamma(y,z) &= \text{second smallest element of } \{\min B_1, \ldots, \min B_r\} \\ &= \text{smallest block minimum in } y \text{ that is not a block} \\ &\ \text{minimum in } z \\ &= \text{smallest edge label of } [y,z]_{\Pi_n} \text{ under the edge-labelling } \delta. \end{aligned} \tag{2}$$

See Figure 5(b) for examples. Note that the label set for γ is $\{2, 3, \ldots, n\}$ and that if $r = 2$, then $\gamma(y,z)$ equals $\delta(y,z)$. We see that the chain

$$\hat{0} < \{1,2\}\{3\}\cdots\{n\} < \{1,2,3\}\{4\}\cdots\{n\} < \cdots < \{1,2,\ldots,n-1\}\{n\} < \hat{1}$$

is an increasing maximal chain in NS_n under γ.

Theorem 5. *The edge-labelling γ is an interpolating EL-labelling for NS_n.*

Applying Theorem 3, we get the following result:

Corollary 5.2. *NS_n is left modular.*

In preparation for proving Theorem 5, we wish to get a firmer grasp on NS_n. Suppose $x, y \in NS_n$. While the meet of x and y in NS_n is just the meet of x and y in Π_n, the situation for joins is more complicated. The next lemma, crucial to the proof that γ is an EL-labelling, helps us to understand important types of joins. From now on, unless otherwise specified, $x \vee y$ with $x, y \in NS_n$ will denote the join of x and y in NS_n. Furthermore, if $l_0 < l_1 < \cdots < l_r$ are block minima in y, then $\langle l_i \rangle$ will denote the block of y with minimum element l_i, and $\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_r \rangle$ will denote the minimum element $z \in NS_n$ for which the elements of $\langle l_0 \rangle, \langle l_1 \rangle, \cdots, \langle l_r \rangle$ are all in a single block. Note that z is well-defined, since it is the meet of all those elements of NS_n that have the required elements in a single block.

Lemma 5.3. *Suppose $l_0 < l_1 < \cdots < l_r$ are block minima in y and that*

$$y \vee (\langle l_0 \rangle \cup \langle l_1 \rangle) = y \vee (\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_r \rangle).$$

Then

$$y \vee (\langle l_i \rangle \cup \langle l_j \rangle) = y \vee (\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_r \rangle).$$

for any $0 \le i < j \le r$.

In words, this says that if merging the blocks $\langle l_0\rangle$ and $\langle l_1\rangle$ in y requires us to merge all of $\langle l_0\rangle, \langle l_1\rangle, \ldots, \langle l_r\rangle$, then merging any two of these blocks also requires us to merge all of them.

Proof. The proof is by induction on r, with the result being trivially true when $r = 1$. While elementary, the details are a little intricate. To gain a better understanding, the reader may wish to treat the proof as an exercise. If $i < j < r - 1$, then by the induction assumption and the hypothesis that $y \vee (\langle l_0\rangle \cup \langle l_1\rangle) = y \vee (\langle l_0\rangle \cup \langle l_1\rangle \cup \cdots \cup \langle l_r\rangle$, we have

$$y \vee (\langle l_i\rangle \cup \langle l_j\rangle) = y \vee (\langle l_0\rangle \cup \langle l_1\rangle \cup \cdots \cup \langle l_{r-1}\rangle) = y \vee (\langle l_0\rangle \cup \langle l_1\rangle \cup \cdots \cup \langle l_r\rangle),$$

as required. Therefore, it suffices to let $j = r$.

Since $y \vee (\langle l_0\rangle \cup \langle l_1\rangle) = y \vee (\langle l_0\rangle \cup \langle l_1\rangle \cup \cdots \cup \langle l_{r-1}\rangle) = y \vee (\langle l_0\rangle \cup \langle l_1\rangle \cup \cdots \cup \langle l_r\rangle)$, we know that $\langle l_0\rangle \cup \langle l_1\rangle \cup \cdots \cup \langle l_{r-1}\rangle$ forms a straddle with $\langle l_r\rangle$. There are two ways in which this might happen.

Suppose we have $a < b < c < d$ with $a, d \in \langle l_0\rangle \cup \langle l_1\rangle \cup \cdots \cup \langle l_{r-1}\rangle$ and $b, c \in \langle l_r\rangle$. Suppose $d \in \langle l_s\rangle$ in y. Then, since $l_s < l_r \leq b < c$, we have that $l_s < b < c < d$ is a straddle in y, which contradicts $y \in NS_n$.

Secondly, suppose we have $a < b < c < d$ with $a, d \in \langle l_r\rangle$ and $b, c \in \langle l_0\rangle \cup \langle l_1\rangle \cup \cdots \cup \langle l_{r-1}\rangle$. Suppose $b \in \langle l_s\rangle$ and $c \in \langle l_t\rangle$. Now $c > b > a \geq l_r > l_s, l_t$. If $s = t$ then y has a straddle, so we can assume that $l_s \neq l_t$ and that $l_i \neq l_t$, with the argument being similar if $l_i \neq l_s$. If $l_i < l_t$, then $l_i < l_t < c < d$ is a straddle when we merge blocks $\langle l_i\rangle$ and $\langle l_r\rangle$ in y. Therefore,

$$y \vee (\langle l_i\rangle \cup \langle l_r\rangle) = y \vee (\langle l_i\rangle \cup \langle l_t\rangle \cup \langle l_r\rangle) = y \vee (\langle l_0\rangle \cup \langle l_1\rangle \cup \cdots \cup \langle l_r\rangle) \tag{3}$$

by the induction assumption. If $l_i > l_t$, then $l_t < l_i < l_r < c$ is a straddle when we merge blocks $\langle l_i\rangle$ and $\langle l_r\rangle$ in y, also implying (3). ☐

Lemma 5.4. *Suppose $y < z$ in NS_n and that $[y, z]_{\Pi_n}$ has edge labels $l_1 < l_2 < \cdots < l_s$ under the edge-labelling δ.*

(i) *There is exactly one edge of the form $y \lessdot w$ with $\gamma(y, w) = l_1$ in $[y, z]_{NS_n}$.*

(ii) *On any unrefinable chain $y \lessdot u_0 \lessdot u_1 \lessdot \cdots \lessdot u_k = z$ in NS_n, the label l_1 has to appear.*

Proof. (i) We first prove the existence of w. Let l_0 be the minimum of the block of z containing l_1 and set $w = y \vee (\langle l_0\rangle \cup \langle l_1\rangle)$. Suppose $y < u \leq w$. We know w is obtained from y by merging the blocks $\langle l_0\rangle, \langle l_1\rangle, \langle l_{i_1}\rangle, \langle l_{i_2}\rangle, \langle l_{i_r}\rangle$, for some $0 \leq r < s$. Applying Lemma 5.3, we get that $u = w$ and so $y \lessdot w$. By definition of γ, we have that $\gamma(y, w) = l_1$.

It remains to prove uniqueness. Suppose $w' \in NS_n$ with $y \lessdot w'$ in $[y, z]$. If $\gamma(y, w') = l_1$, then we see that the blocks $\langle l_0\rangle$ and $\langle l_1\rangle$ must be merged in w'. Therefore, these two blocks are merged in $w \wedge w'$, which is thus greater than y. Since $y \lessdot w, w'$, we conclude that $w = w'$.

(ii) Consider the chain $y = u_0 < u_1 < \cdots < u_k = z$ as a chain in Π_n. Since δ is an S_{n-1} EL-labelling for Π_n (once we subtract 1 from every label), the label l_1 has to appear on every maximal chain of $[y, z]_{\Pi_n}$. It particular, it has to appear in one of the intervals $[u_i, u_{i+1}]_{\Pi_n}$ for $0 \leq i < k$. Therefore, by (2), we get that $\gamma(u_i, u_{i+1}) = l_1$ for some $0 \leq i < k$. ☐

Proposition 5.5. *The edge-labelling γ is an EL-labelling for NS_n.*

Proof. Consider $y, z \in NS_n$ with $y < z$. Suppose $[y,z]_{\Pi_n}$ has edge labels $l_1 < l_2 < \cdots < l_s$. By (2), these are the only edge labels that can appear in $[y,z]_{NS_n}$. We now describe a recursive construction of an unrefinable chain $\lambda : y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_k = z$ in NS_n. We let w_1 be the w of Lemma 5.4, i.e. w_1 is that unique element of the interval $[y,z]$ in NS_n that covers y and satisfies $\gamma(y, w_1) = l_1$. Obviously, the labels in the interval $[w_1, z]$ are all greater than l_1. Now we apply the same argument in the interval $[w_1, z]$ to define w_2 and repeat until we have constructed all of λ. Clearly, λ is then an increasing chain. By Lemma 5.4(i), it has the lexicographically least set of labels. By Lemma 5.4(ii), it is the only increasing chain from y to z. □

Proof of Theorem 5. Suppose we have $y \lessdot u \lessdot z$ in NS_n with $\gamma(y,u) > \gamma(u,z)$. Let $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_k = z$ be the unique increasing chain of $[y,z]$ in NS_n. By Lemma 5.4, we know that $\gamma(w_0, w_1) = \gamma(u,z) = l_1$, the smallest edge label of $[y,z]_{\Pi_n}$.

To show that $\gamma(y,u) = \gamma(w_{k-1}, w_k)$, we have to work considerably harder. We will continue to write $\langle m \rangle$ to denote the block of y whose minimum is m and we suppose that u is obtained from y by merging blocks $\langle m_0\rangle, \langle m_1\rangle, \ldots, \langle m_s\rangle$ of y, with $m_0 < m_1 < \cdots < m_s$. We will write $\langle l\rangle_u$ to denote the block of u whose minimum is l, and we suppose that z is obtained from u by merging blocks $\langle l_0\rangle_u, \langle l_1\rangle_u, \ldots, \langle l_r\rangle_u$, with $l_0 < l_1 < \cdots < l_r$. With the structure of the chain $y \lessdot u \lessdot z$ thus fixed, we now can deduce information about the structure of the increasing chain.

If l_0 and m_0 are distinct and are both block minima in z, then all the l_i's and m_j's are distinct. It follows that z is obtained from y by merging blocks $\langle l_0\rangle, \langle l_1\rangle, \ldots, \langle l_r\rangle$ and separately merging blocks $\langle m_0\rangle, \langle m_1\rangle, \ldots, \langle m_s\rangle$. Since $y \lessdot u \lessdot z$, we get that $k = 2$ and $\gamma(y,u) = \gamma(w_1, z) = m_1$. We assume, therefore, that $m_0 = l_i$ for some $0 \le i \le r$.

As usual, we let $w_1 = y \vee (\langle l_0\rangle \cup \langle l_1\rangle)$. Now consider

$$w = y \vee \Big(\bigcup_{i:\, l_i < m_1} \langle l_i \rangle \Big).$$

Since $l_1 < m_1$, we know that $w \ge w_1$. Let B denote the the block of w containing all $\langle l_i\rangle$ satisfying $l_i < m_1$. Since $m_0 < m_1$, we know that $m_0 \in B$. In fact, if we can show that $m_1 \notin B$, then we can now complete the proof. Indeed, assume $m_1 \notin B$ and let $w' = w \vee (B \cup \langle m_1\rangle)$. Now w' has m_0 and m_1 in the same block and so satisfies $w' \ge u$, since $u = y \vee (\langle m_0\rangle \cup \langle m_1\rangle)$. Also, w' has l_0 and l_1 in the same block and so satisfies $w' \ge z$, since $z = u \vee (\langle l_0\rangle \cup \langle l_1\rangle)$. Hence, $w' = z$. By Lemma 5.3 (substitute w for y, and $m_1 < \cdots < m_s$ for $l_1 < \cdots < l_r$), we see that $w \lessdot w'$. Now $\gamma(w, w') = m_1$, while the edge labels of $[y,w]_{\Pi_n}$ all come from the set $\{l_i \mid l_i < m_1\}$, implying that w is on the increasing chain between y and z. Therefore, $w = w_{k-1}$ and so $\gamma(w_{k-1}, w_k) = \gamma(y,u)$.

It remains to show that $m_1 \notin B$. In fact, we will show that $m_j \notin B$ for any $j \ge 1$. Consider the set:

$$\tilde{B} = \bigcup_{i:\, l_i < m_1} \langle l_i \rangle$$

We will show that $\tilde{B}$ does not form a straddle with any $\langle m_j\rangle$ for $j \ge 1$. From that, it follows immediately that $B = \tilde{B}$, and therefore that $m_1 \notin B$, as desired.

For $j \ge 1$, if $\langle m_j\rangle$ is a singleton, then $\tilde{B}$ does not form a straddle with $\langle m_j\rangle$. So suppose that $|\langle m_j\rangle| \ge 2$. Let m'_j denote the second smallest element of $\langle m_j\rangle$. Observe the following:

- If $\langle m_0 \rangle$ contains an element greater than m'_j, then $\langle m_0 \rangle$ and $\langle m_j \rangle$ form a straddle in y, which is impossible.
- If $\langle m_0 \rangle$ has more than one element between m_j and m'_j, then we can draw the same conclusion.
- Consider those $l_i < m_1$ with $l_i \neq m_0$. If $\langle l_i \rangle$ contains an element greater than m_j, then $\langle l_i \rangle_u$ forms a straddle in u with $\langle m_0 \rangle \cup \langle m_1 \rangle \cup \cdots \cup \langle m_s \rangle$, which is impossible.

Combining these three observations, we see that $\tilde{B}$ contains no elements greater than m'_j, and at most one element between m_j and m'_j. In particular, it does not form a straddle with $\langle m_j \rangle$, as desired. ☐

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