COMMUTATION AND NORMAL ORDERING FOR OPERATORS ON SYMMETRIC FUNCTIONS

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Dedicated to Ira Gessel on the occasion of his retirement.

Abstract. We study the commutation relations and normal ordering between families of operators on symmetric functions. These operators can be naturally defined by the operations of multiplication, Kronecker product, and their adjoints. As applications we give a new proof of the skew Littlewood–Richardson rule and prove an identity about the Kronecker product with a skew Schur function.

1. Introduction

The theory of symmetric functions is rich in identities involving Schur functions. The ubiquity of Schur functions is due, among other reasons, to their roles in representation theory, Schubert calculus, and to their beautiful combinatorial definition.

Some of these identities admit an elegant interpretation as a "normal ordering" relations between pairs of operators related to Schur functions. An example is the following identity due to Foulkes:

$$D_\beta(fg) = \sum_{\lambda,\mu} c^{\beta}_{\lambda,\mu} D_\lambda(f) D_\mu(g),$$

where D_β is the adjoint of the operator U_β of multiplication by the Schur function s_β, and the $c^{\beta}_{\lambda,\mu}$ are the Littlewood–Richardson coefficients. As we will see, Foulkes' Identity can be rewritten as

$$D_\beta U_\alpha = \sum_{\mu,\nu} \left(\sum_{\lambda} c^{\alpha}_{\lambda,\mu}\, c^{\beta}_{\lambda,\nu} \right) U_\mu D_\nu \,.$$

Here, we are given two families of operators, and we decompose their products as linear combinations of products of operators in the reverse order. Just like in the normal ordering problem for creation and annihilation operators in physics.

In this paper, we consider six normal ordering relations (Theorem 3.1 and Corollary 3.2) between pairs of operators involving: U_λ, D_λ, and the operators K_λ, that corresponds to the Kronecker product with the Schur function s_λ, and $\overline{K}_\lambda$, that map any homogeneous symmetric function f of degree n to its Kronecker product with $s_{(n-|\lambda|,\lambda)}$. Note that the operator $\overline{K}_\lambda$ has been previously studied in [6] and [15].

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The operators $U_{(1)}$ and $D_{(1)}$ are known to satisfy the commutation relation $D_{(1)}U_{(1)} = U_{(1)}D_{(1)} + 1$. In [14], Stanley built on this relation to define *differential posets*. Later on, Ira Gessel [5] found the commutation relations for the operators D_β, and U_α indexed by one row and one column partitions. We generalize Gessel's commutation relations, and obtain that

$$[D_\beta, U_\alpha] = \sum_{\lambda \neq (0)} U_{\alpha/\lambda} D_{\beta/\lambda} = \sum_{\lambda \neq (0)} (-1)^{|\lambda|-1} D_{\beta/\lambda'} U_{\alpha/\lambda}.$$

In [3], Fulman demonstrates that these sorts of commutation relations are also well suited for analyzing the convergence rate of certain Markov chains.

The proofs presented in this article build on the generating functions of the considered families of operators, and operations on alphabets ("λ–ring formalism"). This approach gives us a powerful and uniform tool that we hope clarifies the source of these relations.

We present two applications of our identities. The first one is a proof of the *skew Littlewood–Richardson Rule*, a combinatorial rule that gives the product of two skew Schur functions as a linear combination of skew Schur functions, based on counting Young tableaux (Theorem 7.3). This rule was conjectured in [1], and proved in [8]. Our proof relies on the normal ordering relation that decomposes the products $U_\alpha D_\beta$ as linear combinations of products of the form $D_{\beta/\lambda'} U_{\alpha/\lambda}$. It generalizes the algebraic proof given by Thomas Lam of the skew Pieri Rule (a particular case of the skew Littlewood–Richardson Rule) in the appendix of [1]. Indeed, Lam's proof relies on the same normal ordering relation, in the particular case of β having only one part.

The second application exploits our normal ordering relation for the products $\overline{K}_1 D_\lambda$. We extend the combinatorial rule for the expansion in the Schur basis of the Kronecker product of $s_{(n-1,1)}$ with a Schur function, to the Kronecker product of $s_{(n-1,1)}$ with any skew Schur function (Theorem 8.1). Additionally, we give a different, combinatorial, proof of the same result. In [4], Ira Gessel considered the Kronecker product between Schur functions indexed by zig-zag shapes, that are particular instances of skew shapes.

The remainder of the paper is structured as follows. In Section 2, we present necessary background on the operators U_λ, D_λ, K_λ and $\overline{K}_\lambda$. In Section 3, we present the statements of our six normal ordering relations, along with their connections to each other and to results in the literature. The proofs of these six identities, and the necessary background for the proofs, appear in the first five subsections of Section 4. In Section 5, we study whether our identities give the unique way of performing the normal orderings, while in Section 6, we show that $\overline{K}_f$ is in the algebra generated by U_g and D_h. Section 7 contains our first application, specifically a new proof of the skew Littlewood–Richardson rule, along with an algebraic derivation of the algebraic version of the skew Littlewood–Richardson rule. We close in Section 8 with our second application, which is a combinatorial expression for the Kronecker product of a general skew Schur function and $s_{(n-1,1)}$.

2. Preliminaries

We follow Macdonald's book [11] as our main reference for the theory of symmetric functions, with the exception that we draw our Young diagrams the French way. For the approach to symmetric functions using operators on alphabets we follow Lascoux's book [9], and the work of Thibon et al. See [13], [15] and [16].

Let *Sym* denote the algebra of symmetric functions with rational coefficients, with respect to ordinary multiplication, and equipped with the scalar product, $\langle \mid \rangle$, that makes the Schur functions orthonormal. We also consider a second product on *Sym*, the Kronecker product, defined on the power sum symmetric functions, p_λ, by $p_\lambda * p_\mu = z_\lambda \delta_{\lambda,\mu} p_\lambda$, where $\delta_{\lambda,\mu}$ is the Kronecker delta and $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$ with m_i the multiplicity of i as a part of λ. For more details, see [11, p. 115–116].

For any symmetric function f, let U_f denote the operator defined by multiplication by f, and let K_f be the operator defined by Kronecker multiplication by f. That is,

$$U_f(g) = fg, \qquad K_f(g) = f * g.$$

For simplicity, given any family of operators P_f indexed by symmetric functions, we write P_λ (respectively $P_{\lambda/\mu}$) instead of P_{s_λ} (resp. $P_{s_{\lambda/\mu}}$) when the indexing function is a Schur function s_λ (resp. a skew Schur function $s_{\lambda/\mu}$). If μ is not contained in λ, then $P_{\lambda/\mu}$ is multiplication by zero.

In this setting, Pieri's rule is the statement $U_{(n)}(s_\lambda) = \sum s_{\widehat{\lambda}}$, where we sum over all $\widehat{\lambda}$ such that $\widehat{\lambda}/\lambda$ is an *n-horizontal strip*, meaning that $\widehat{\lambda}/\lambda$ is a skew shape with n boxes having at most one box in each column. On the other hand, $K_{(n)}(s_\lambda) = s_\lambda$ if $\lambda \vdash n$, and zero otherwise.

More generally,

$$U_\mu(s_\nu) = \sum_{\lambda \vdash |\mu|+|\nu|} c^\lambda_{\mu,\nu} s_\lambda, \qquad K_\mu(s_\nu) = \sum_{\lambda \vdash |\mu|} g_{\mu,\nu,\lambda} s_\lambda$$

where the coefficients $c^\lambda_{\mu,\nu}$ are the Littlewood–Richardson coefficients, and $g_{\mu,\nu,\lambda}$ are the Kronecker coefficients.

Let D_f be the adjoint operator to U_f. This operator is denoted by $f^\perp$ in Macdonald's book. The operator D_λ is the "skewing" operator: $D_\lambda(s_\mu) = s_{\mu/\lambda}$ for all μ.

Finally, we introduce the operator $\overline{K}_f$. Letting λ be a partition and g be any homogeneous symmetric function of degree n, we define

$$\overline{K}_\lambda(g) = s_{(n-|\lambda|,\lambda)} * g\,.$$

Here $(n-|\lambda|,\lambda) = (n-|\lambda|,\lambda_1,\lambda_2,\ldots)$ is a sequence of integers, but it is not necessarily decreasing when n is small. We define $s_{(n-|\lambda|,\lambda)}$ by means of the Jacobi–Trudi formula:

$$s_{(\alpha_1,\alpha_2,\ldots,\alpha_N)} = \det(h_{\alpha_i+j-i})_{i,j=1\ldots N}$$

where h_k are the complete homogeneous sums for $k>0$, $h_0 = 1$ and $h_k = 0$ when $k<0$. This determinant coincides with the Schur function s_α when α is weakly decreasing (*i.e.,* is a partition) but makes sense perfectly even when α is not. Once the operators $\overline{K}_\lambda$ are defined, the definition is extended by linearity to $\overline{K}_f$ for any symmetric function f.

The operators $\overline{K}_f$ have a basis-free definition given by $\overline{K}_f = K_{\Gamma_1 f}$ where Γ_1 is the *vertex operator*

$$\Gamma_1 = \left(\sum_{i=0}^{\infty} U_{(i)}\right)\left(\sum_{j=0}^{\infty}(-1)^j D_{(1^j)}\right) \tag{2.1}$$

considered in [15, p. 211] and in [13, §3] to study properties of stability of Kronecker and plethysm coefficients.

A well-known case of $\overline{K}_\lambda$ is when $\lambda = (1)$. We pause to describe an identity that we will generalize in Section 8 using the normal ordering relations. A *corner* of α is a box of α whose removal results in another partition, and we denote by #corners(α) the number of corners of α. Denote by α^- the set of partitions that result from removing a corner of α. Similarly, α^+ will denote the set of those partitions β such that $\alpha \in \beta^-$. We use $\alpha^\mp$ to denote the set of partitions not equal to α that can be obtained by removing a corner of α and then adding a box to the result. Equivalently, $\alpha^\mp$ is the set of partitions that can be obtained from α by first adding a box and then removing a different box. For example, 31 has two corners, and $(31)^\mp = \{(4), (22), (211)\}$. We begin Subsection 6 with a short proof that

$$\overline{K}_{(1)} s_\alpha = (\#\mathrm{corners}(\alpha) - 1) s_\alpha + \sum_{\beta \in \alpha^\mp} s_\beta \,. \tag{2.2}$$

Chauve and Goupil [6] used (2.2) to give a combinatorial interpretation for $(\overline{K}_{(1)})^k(s_{(1)})$ in terms of oscillating tableaux. In Section 6, we give a natural extension of (2.2) to the case when s_α is replaced by a general skew Schur function $s_{\alpha/\theta}$ (see Theorem 8.1).

We remark that the operators K_f and $\overline{K}_f$ are self-adjoints, a fact that can be easily checked using the power sum basis.

3. Statement of the results

The main result of the paper is a list of six normal ordering relations, which we state below in Theorem 3.1 and Corollary 3.2 in two slightly different ways. These relations will be proved in Section 4.

Theorem 3.1. *For any partitions α and β we have the following identities (where λ, τ and ν each run over the set of all partitions).*

$$D_\beta U_\alpha = \sum_\lambda U_{\alpha/\lambda} D_{\beta/\lambda} \tag{3.1}$$

$$U_\alpha D_\beta = \sum_\lambda (-1)^{|\lambda|} D_{\beta/\lambda'} U_{\alpha/\lambda} \tag{3.2}$$

$$K_\beta U_\alpha = \sum_\lambda U_{s_{\beta/\lambda} * s_\alpha} K_\lambda \tag{3.3}$$

$$D_\alpha K_\beta = \sum_\lambda K_\lambda D_{s_{\beta/\lambda} * s_\alpha} \tag{3.4}$$

$$\overline{K}_\beta U_\alpha = \sum_{\tau,\nu} U_{(s_{\beta/\nu} * s_\tau) s_{\alpha/\tau}} \overline{K}_\nu \tag{3.5}$$

$$D_\alpha \overline{K}_\beta = \sum_{\tau,\nu} \overline{K}_\nu D_{(s_{\beta/\nu} * s_\tau) s_{\alpha/\tau}} \tag{3.6}$$

Corollary 3.2. *For any partitions α and β we have the following identities.*

$$D_\beta U_\alpha = \sum_{\mu,\nu}\left(\sum_\lambda c^\alpha_{\lambda,\mu}\, c^\beta_{\lambda,\nu}\right) U_\mu D_\nu$$

$$U_\alpha D_\beta = \sum_{\mu,\nu}\left(\sum_\lambda (-1)^{|\lambda|} c^\alpha_{\lambda,\mu}\, c^\beta_{\lambda',\nu}\right) D_\nu U_\mu$$

$$K_\beta U_\alpha = \sum_{\mu,\nu}\left(\sum_\lambda g_{\alpha,\lambda,\mu}\, c^\beta_{\lambda,\nu}\right) U_\mu K_\nu$$

$$D_\alpha K_\beta = \sum_{\mu,\nu}\left(\sum_\lambda g_{\alpha,\lambda,\mu}\, c^\beta_{\lambda,\nu}\right) K_\nu D_\mu$$

$$\overline{K}_\beta U_\alpha = \sum_{\mu,\nu}\left(\sum_{\lambda,\sigma,\tau,\theta} g_{\lambda,\tau,\theta}\, c^\beta_{\lambda,\nu}\, c^\alpha_{\tau,\sigma}\, c^\mu_{\theta,\sigma}\right) U_\mu \overline{K}_\nu$$

$$D_\alpha \overline{K}_\beta = \sum_{\mu,\nu}\left(\sum_{\lambda,\sigma,\tau,\theta} g_{\lambda,\tau,\theta}\, c^\beta_{\lambda,\nu}\, c^\alpha_{\tau,\sigma}\, c^\mu_{\theta,\sigma}\right) \overline{K}_\nu D_\mu$$

Identities (3.1) and (3.3) are avatars of well-known identities of Foulkes and Littlewood. Indeed, if we apply the operators in (3.1) and (3.3) to the Schur function s_γ we get

$$D_\beta(s_\alpha s_\gamma) = \sum_\lambda s_{\alpha/\lambda} D_{\beta/\lambda}(s_\gamma), \tag{3.7}$$

$$s_\beta * (s_\alpha s_\gamma) = \sum_\lambda (s_{\beta/\lambda} * s_\alpha)(s_\lambda * s_\gamma). \tag{3.8}$$

By linearity, we can replace s_α and s_γ with arbitrary symmetric functions f and g. Also if we expand $s_{\beta/\lambda} = \sum_\mu c^\beta_{\lambda,\mu} s_\mu$, we obtain

$$D_\beta(fg) = \sum_{\lambda,\mu} c^\beta_{\lambda,\mu} D_\lambda(f) D_\mu(g), \tag{3.9}$$

$$s_\beta * (fg) = \sum_{\lambda,\mu} c^\beta_{\lambda,\mu} (s_\mu * f)(s_\lambda * g). \tag{3.10}$$

Formula (3.9) was obtained by Foulkes ([2, §3.b] , also mentioned in [11, I.§5 Ex. 25.(d)], while (3.10) is due to Littlewood ([10, Theorem III], see also [11, I.§7 Ex. 23.(c)]).

The similarity between the two identities (3.9) and (3.10) is explained in [15, Proposition 6.4] and [16]. They are both particular cases of a general splitting formula for spaces endowed with a product and a coproduct that are compatible.

Formula (3.2) can be stated as

$$s_\alpha s_{\gamma/\beta} = \sum_\lambda (-1)^{|\lambda|} D_{\beta/\lambda'}(s_{\alpha/\lambda} s_\gamma). \tag{3.11}$$

Formula (3.2) happens to be closely related to the *Skew Littlewood–Richardson Rule*; see Section 7. In [8, Lemma 1.1] Lam, Lauve, and Sottile obtain a more general version of Formula (3.2) valid for arbitrary pairs of dual Hopf algebras.

As mentioned in the introduction, Ira Gessel, [5], established special cases of (3.1) and (3.2) when the Schur functions are indexed by one-row or one-column shapes. He showed that

$$\begin{aligned} D_n U_m &= \sum_i U_{m-i} D_{n-i}\,, \\ U_m D_n &= D_n U_m - D_{n-1} U_{m-1}\,, \\ D_{(1^n)} U_m &= U_m D_{(1^n)} + U_{m-1} D_{(1^{n-1})}. \end{aligned}$$

Since K_β and $\overline{K}_\beta$ are self-adjoint, and U_α and D_α are adjoint of each other, (3.4) and (3.6) are obtained from (3.3) and (3.5) respectively by taking adjoints.

An interesting and elegant way of stating some of the results of Theorem 3.1 is in terms of commutators.

Corollary 3.3. *For any two partitions α and β, we have*

$$\begin{aligned} [D_\beta, U_\alpha] &= \sum_{\lambda \neq (0)} U_{\alpha/\lambda} D_{\beta/\lambda} = \sum_{\lambda \neq (0)} (-1)^{|\lambda|-1} D_{\beta/\lambda'} U_{\alpha/\lambda}\,, \\ [\overline{K}_\beta, U_\alpha] &= \sum_{(\tau,\nu) \neq ((0),\beta)} U_{(s_{\beta/\nu} * s_\tau) s_{\alpha/\tau}} \overline{K}_\nu\,, \\ [D_\alpha, \overline{K}_\beta] &= \sum_{(\tau,\nu) \neq ((0),\beta)} \overline{K}_\nu D_{(s_{\beta/\nu} * s_\tau) s_{\alpha/\tau}}\,, \end{aligned}$$

where (0) *denotes the empty partition.*

4. Proof of the identities

In this section, we prove all six identities of Theorem 3.1. We begin with an overview of the method of proof. Let P be any of the families of operators U, D, K and $\overline{K}$. Introduce an auxiliary alphabet A. The Schur generating series of P is defined as

$$\sum_\lambda P_\lambda s_\lambda[A].$$

This can be interpreted as the linear map that sends any symmetric function g to the expression

$$\sum_\lambda P_\lambda(g) s_\lambda[A].$$

For each of the four operators under consideration, the effect of the Schur generating function operator is described nicely by means of operations on alphabets (Lemma 4.2). The identities of Theorem 3.1 are derived at the level of generating series. The result is then recovered by extracting coefficients by means of the appropriate scalar products.

4.1. **Preliminary: operations on alphabets.** Let $X = \{x_1, x_2, \ldots\}$ be the underlying alphabet for the symmetric functions in *Sym*. Any infinite alphabet A gives rise to a copy *Sym*(A) of *Sym*. In this copy, the corresponding scalar product will be denoted by $\langle\ |\ \rangle_A$ and the element corresponding to $f \in Sym$, by $f[A]$. Accordingly, the scalar product $\langle\ |\ \rangle$ of *Sym* and elements $f \in Sym$ will be denoted sometimes by $\langle\ |\ \rangle_X$ and $f[X]$.

If A and B are two alphabets, the tensor product $Sym(A) \otimes Sym(B)$ is endowed with the induced scalar product $\langle\ |\ \rangle_{A.B}$.

Both the Kronecker product $*$ and the adjoint D_f of the operator of multiplication by a symmetric function f will be only considered with respect to $Sym = Sym(X)$.

The map $f \mapsto f[A]$ is a specialization. In particular, it is a morphism of algebras from *Sym* to *Sym*(A). It is convenient to write again $f \mapsto f[A]$ (rather than $A(f)$) for any *morphism of algebras* from *Sym* to some commutative algebra $\mathcal{A}$, and consider it as a "specialization at the virtual alphabet A."

Since the power sum symmetric functions p_k ($k \geqslant 1$) generate *Sym* and are algebraically independent, the map

$$A \mapsto (p_1[A], p_2[A], \ldots) \tag{4.1}$$

is a bijection from the set of all morphisms of algebras from *Sym* to $\mathcal{A}$ to the set of infinite sequences of elements from $\mathcal{A}$. This set of sequences is endowed with its operations of component-wise sum, product, and multiplication by a scalar. The bijection (4.1) is used to lift these operations to the set of morphisms from *Sym* to $\mathcal{A}$. This defines expressions like $f[A+B]$ and $f[AB]$, where f is a symmetric function and A and B are two "virtual alphabets," and more general expressions $f[P(A, B, \ldots)]$ where $P(A, B, \ldots)$ is a polynomial in several virtual alphabets A, B ... with coefficients in the basis field. Note that, by definition, for any power sum p_k ($k \geqslant 1$), virtual alphabets A and B, and scalar z,

$$p_k[A+B] = p_k[A] + p_k[B], \qquad p_k[AB] = p_k[A] \cdot p_k[B], \qquad p_k[zA] = z\, p_k[A].$$

In our calculations below, the morphism $f \mapsto f[1]$ will appear: it is the specialization at $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ .... The morphism $f \mapsto f[X]$ is just the identity of *Sym*.

Let σ be the generating series for the complete homogeneous symmetric functions h_n, meaning $\sigma = \sum_{n=0}^{\infty} h_n$, where $h_0 = 1$. Recall from [11, I.§2] that we also have

$$\sigma = \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{k}\right) = \sum_{\lambda} \frac{p_\lambda}{z_\lambda}$$

where the last sum is carried over all partitions λ.

We will make use of the following identities.

Lemma 4.1. *Let A and B be any two alphabets, and f and g be any two symmetric functions. Then we have the following identities.*

$$\sigma[A+B] = \sigma[A]\sigma[B] \tag{4.2}$$

$$\sigma[AB] = \sum_{\lambda} s_\lambda[A] s_\lambda[B] \qquad \textit{(Cauchy Identity)} \tag{4.3}$$

$$\sigma[-AB] = \sum_{\lambda} (-1)^{|\lambda|} s_{\lambda'}[A] s_\lambda[B] \tag{4.4}$$

$$D_{\sigma[AX]}(f[X]) = f[X+A] \tag{4.5}$$

$$\sigma[AX] * f[X] = f[AX] \tag{4.6}$$

$$\langle\, f[XA] \,|\, g[X] \,\rangle_X = (f * g)[A] \tag{4.7}$$

$$\langle\, \sigma[AB] \,|\, g[B] \,\rangle_B = g[A] \qquad \textit{(Reproducing Kernel)} \tag{4.8}$$

Proof. The identity $\sigma = \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{k}\right)$ already proves (4.2), since $p_k[A+B] = p_k[A] + p_k[B]$ for all k.

Property (4.6) is simply checked on power sums, using the fact that $p_\lambda * p_\mu = \delta_{\lambda,\mu} z_\mu p_\mu$. We have

$$\begin{aligned}\sigma[AX] * p_\mu[X] &= \sum_\lambda \frac{p_\lambda[AX]}{z_\lambda} * p_\mu[X] \\ &= \sum_\lambda \frac{p_\lambda[A]}{z_\lambda} p_\lambda[X] * p_\mu[X] \\ &= p_\mu[A]\, p_\mu[X] = p_\mu[AX].\end{aligned}$$

Property (4.7) is also checked on the basis of power sums; take $f = p_\lambda$ and $g = p_\mu$. Then

$$\langle\, p_\lambda[AX] \,|\, p_\mu[X] \,\rangle_X = \langle\, p_\lambda[A] p_\lambda[X] \,|\, p_\mu[X] \,\rangle_X = p_\lambda[A] \delta_{\lambda,\mu} z_\lambda,$$

which is equal to $(p_\lambda * p_\mu)[A]$.

The Reproducing Kernel property (4.8) is a direct consequence of (4.7) and (4.6). Indeed, in (4.7) take $f = \sigma$ to get $\langle\, \sigma[XA] \,|\, g[X] \,\rangle_X = (\sigma * g)[A]$. In (4.6) take $A = 1$ and replace f with g to get $\sigma * g = g$. This yields $\langle\, \sigma[XA] \,|\, g[X] \,\rangle_X = g[A]$. Then (4.8) is this identity with B instead of X.

The series $\sigma[AB]$ expands as $\sigma[AB] = \sum_\lambda c_\lambda[A] s_\lambda[B]$ where the $c_\lambda[A]$ are symmetric functions in A. Therefore $\langle\, \sigma[AB] \,|\, s_\lambda[B] \,\rangle_B = c_\lambda[A]$. But by the Reproducing Kernel Property, this is also equal to $s_\lambda[A]$. This proves the Cauchy identity (4.3).

The dual Cauchy identity is obtained by replacing A with $-A$, and using the fact that $s_\lambda[-A] = (-1)^{|\lambda|} s_{\lambda'}[A]$ (see [11, I. (3.10)]).

Finally, (4.5) is a direct consequence of Taylor's formula for polynomials:

$$F(x_1+y_1, x_2+y_2, \ldots) = \sum_{\alpha_1,\alpha_2,\ldots=0}^{\infty} \frac{\partial^{\alpha_1} F}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2} F}{\partial x_1^{\alpha_2}} \cdots \frac{y_1^{\alpha_1}}{\alpha_1!} \frac{y_2^{\alpha_2}}{\alpha_2!} \cdots = \exp\left(\sum_{i=1}^{\infty} y_i \frac{\partial}{\partial x_i} \right) F\,.$$

Indeed, let f be any symmetric function. Let F be the unique polynomial such that $f = F(p_1, p_2, \ldots)$. Plugging in $p_i[X]$ for x_i, and $p_i[A]$ for y_i in the above identity, we get

$$f[X+A] = \exp\left(\sum_{i=1}^{\infty} p_i[A] \frac{\partial}{\partial p_i[X]} \right) f[X] = \exp\left(\sum_{i=1}^{\infty} \frac{p_i[A] D_{p_i[X]}}{i} \right) f[X]\,,$$

since $D_{p_k} = k \frac{\partial}{\partial p_k}$ for any $k > 0$. Now,

$$\sigma[AX] = \exp\left(\sum_{i=1}^{\infty} \frac{p_i[A] p_i[X]}{i} \right).$$

Therefore,

$$D_{\sigma[AX]}(f[X]) = \exp\left(\sum_{i=1}^{\infty} \frac{p_i[A] D_{p_i}}{i} \right) f[X] = f[X+A].$$

□

4.2. **Generating series.** Let P be any of the families of operators U, D, K and $\overline{K}$. Introduce an auxiliary alphabet A and the following generating series for P:

$$\sum_\lambda P_\lambda s_\lambda[A].$$

We use the linearity of $f \mapsto P_f$ to simplify this expression as follows:

$$\sum_\lambda P_\lambda s_\lambda[A] = \sum_\lambda P_{s_\lambda[X]} s_\lambda[A] = P_{\sum_\lambda s_\lambda[X] s_\lambda[A]} = P_{\sigma[AX]}$$

by the Cauchy identity (4.3).

Note that any operator P_λ can be recovered from the generating series with a coefficient extraction by means of a scalar product:

$$P_\lambda(f) = \left\langle P_{\sigma[AX]}(f[X]) \,\middle|\, s_\lambda[A] \right\rangle_A .$$

The generating series $P_{\sigma[AX]}$ also acts linearly on symmetric functions. The following lemma describes the effect of all four generating series $U_{\sigma[AX]}$, $D_{\sigma[AX]}$, $K_{\sigma[AX]}$ and $\overline{K}_{\sigma[AX]}$.

Lemma 4.2. *Let $f[X]$ be any symmetric function.*

$$U_{\sigma[AX]}(f[X]) = \sigma[AX] \cdot f[X] \tag{4.9}$$

$$D_{\sigma[AX]}(f[X]) = f[X + A] \tag{4.10}$$

$$K_{\sigma[AX]}(f[X]) = f[AX] \tag{4.11}$$

$$\overline{K}_{\sigma[AX]}(f[X]) = \sigma[-A] \cdot f[X(A + 1)] \tag{4.12}$$

Proof. Equation (4.9) is straightforward. Equation (4.10) is (4.5). Equation (4.11) is (4.6). Let us prove (4.12). For any symmetric functions f and g, $\overline{K}_f(g) = \Gamma_1 f * g$, where Γ_1 is the vertex operator defined in (2.1). We will make use of the following identity (see [13, §3]):

$$\Gamma_1 f = \sigma[X] f[X - 1]. \tag{4.13}$$

Therefore, we have

$$\begin{aligned}
\overline{K}_{\sigma[AX]}(f) &= (\Gamma_1\, \sigma[AX]) * f[X] \\
&= (\sigma[X]\, \sigma[A(X-1)]) * f[X] \\
&= \sigma[X + A(X-1)] * f[X] && \text{by (4.2),} \\
&= \sigma[X(A+1) - A] * f[X] \\
&= \sigma[-A]\ (\sigma[X(A+1)] * f[X]) && \text{by (4.2) again,} \\
&= \sigma[-A] \cdot f[X(A+1)] && \text{by (4.6).}
\end{aligned}$$

□

4.3. **Operators U and D.** Let us prove the first two identities in Theorem 3.1, which are (3.1),

$$D_\beta\, U_\alpha = \sum_\lambda U_{\alpha/\lambda} D_{\beta/\lambda}\,,$$

and (3.2),

$$U_\alpha\ D_\beta = \sum_\lambda (-1)^{|\lambda|} D_{\beta/\lambda'} U_{\alpha/\lambda}\,.$$

To this aim, we first establish commutation relations between the generating series of the operators U and D.

Lemma 4.3. *Let A and B be two alphabets. We have*

$$D_{\sigma[BX]}U_{\sigma[AX]} = \sigma[AB]U_{\sigma[AX]}D_{\sigma[BX]}, \tag{4.14}$$

$$U_{\sigma[AX]}D_{\sigma[BX]} = \sigma[-AB]D_{\sigma[BX]}U_{\sigma[AX]}. \tag{4.15}$$

Proof. The second of these identities is obtained straightforwardly from the first after noting that $\sigma[-AB]$ is the inverse of $\sigma[AB]$ (see (4.2)).

Let us prove (4.14) using the properties of Lemma 4.1. We have, for any symmetric function $f[X]$,

$$\begin{aligned}
D_{\sigma[BX]}U_{\sigma[AX]}(f[X]) &= D_{\sigma[BX]}(\sigma[AX]f[X]) \\
&= \sigma[A(X+B)]f[X+B] && \text{by (4.5)}, \\
&= \sigma[AB]\sigma[AX]f[X+B] && \text{by (4.2)}, \\
&= \sigma[AB]\sigma[AX]D_{\sigma[BX]}(f[X]) && \text{by (4.5)}, \\
&= \sigma[AB]U_{\sigma[AX]}D_{\sigma[BX]}(f[X]).
\end{aligned}$$

□

Proof of (3.1). We will use that, since U and $D : Sym \to \mathrm{End}(Sym)$ are morphisms of algebra, we can write, for any symmetric function f, that $U_f = f[U]$ and $D_f = f[D]$. In particular, for the generating series, we have $U_{\sigma[AX]} = \sigma[AU]$ and $D_{\sigma[BX]} = \sigma[BD]$.

In (4.14), the operator $D_\beta U_\alpha$ is the coefficient of $s_\alpha[A]s_\beta[B]$ in the expansion in the Schur basis of $\sigma[AB]U_{\sigma[AX]}D_{\sigma[BX]}$, which is extracted by performing the scalar product with $s_\alpha[A]s_\beta[B]$. Thus

$$\begin{aligned}
D_\beta U_\alpha &= \left\langle\, \sigma[AB]U_{\sigma[AX]}D_{\sigma[BX]} \,\middle|\, s_\alpha[A]s_\beta[B] \,\right\rangle_{A,B} \\
&= \sum_\lambda \left\langle\, s_\lambda[A]s_\lambda[B]U_{\sigma[AX]}D_{\sigma[BX]} \,\middle|\, s_\alpha[A]s_\beta[B] \,\right\rangle_{A,B} && \text{by (4.3)}, \\
&= \sum_\lambda \left\langle\, s_\lambda[A]U_{\sigma[AX]} \,\middle|\, s_\alpha[A] \,\right\rangle_A \left\langle\, s_\lambda[B]D_{\sigma[BX]} \,\middle|\, s_\beta[B] \,\right\rangle_B \\
&= \sum_\lambda \left\langle\, U_{\sigma[AX]} \,\middle|\, s_{\alpha/\lambda}[A] \,\right\rangle_A \left\langle\, D_{\sigma[BX]} \,\middle|\, s_{\beta/\lambda}[B] \,\right\rangle_B \\
&= \sum_\lambda \left\langle\, \sigma[AU] \,\middle|\, s_{\alpha/\lambda}[A] \,\right\rangle_A \left\langle\, \sigma[BD] \,\middle|\, s_{\beta/\lambda}[B] \,\right\rangle_B \\
&= \sum_\lambda s_{\alpha/\lambda}[U]s_{\beta/\lambda}[D] && \text{by (4.8)}, \\
&= \sum_\lambda U_{\alpha/\lambda}D_{\beta/\lambda}.
\end{aligned}$$

□

Identity (3.2) is derived from (4.15) analogously.

4.4. **Operators U and K.** We now turn to the proof of (3.3):

$$K_\beta U_\alpha = \sum_\lambda U_{s_{\beta/\lambda} * s_\alpha} K_\lambda$$

(The identity (3.4) follows straightforwardly from (3.3) by taking adjoints).

Again we consider first commutation relations for the generating series of the operators K and U.

Lemma 4.4. *Let A and B be two alphabets. We have*

$$K_{\sigma[BX]} U_{\sigma[AX]} = U_{\sigma[ABX]} K_{\sigma[BX]}. \tag{4.16}$$

Proof. We have, for any symmetric function f,

$$\begin{aligned} K_{\sigma[BX]} U_{\sigma[AX]}(f[X]) &= K_{\sigma[BX]}(\sigma[AX] f[X]) \\ &= \sigma[ABX] f[BX] && \text{by (4.6)}, \\ &= U_{\sigma[ABX]}(f[BX]) \\ &= U_{\sigma[ABX]} K_{\sigma[BX]}(f[X]) && \text{by (4.6)}. \end{aligned}$$

□

Proof of (3.3). We now get $K_\beta U_\alpha$ from (4.16) by extracting the coefficient of $s_\beta[B] s_\alpha[A]$ in its expansion in terms of Schur functions:

$$\begin{aligned} K_\beta U_\alpha &= \left\langle U_{\sigma[ABX]} K_{\sigma[BX]} \,\middle|\, s_\beta[B] s_\alpha[A] \right\rangle_{A,B} \\ &= \sum_\lambda \left\langle U_{\sigma[ABX]} s_\lambda[B] \,\middle|\, s_\beta[B] s_\alpha[A] \right\rangle_{A,B} K_\lambda \\ &= \sum_\lambda \left\langle U_{\sigma[ABX]} \,\middle|\, s_{\beta/\lambda}[B] s_\alpha[A] \right\rangle_{A,B} K_\lambda \\ &= \sum_\lambda \left\langle \left\langle U_{\sigma[ABX]} \,\middle|\, s_\alpha[A] \right\rangle_A \,\middle|\, s_{\beta/\lambda}[B] \right\rangle_B K_\lambda \\ &= \sum_\lambda \left\langle \left\langle \sigma[ABU] \,\middle|\, s_\alpha[A] \right\rangle_A \,\middle|\, s_{\beta/\lambda}[B] \right\rangle_B K_\lambda \\ &= \sum_\lambda \left\langle s_\alpha[BU] \,\middle|\, s_{\beta/\lambda}[B] \right\rangle_B K_\lambda && \text{by (4.8)}, \\ &= \sum_\lambda (s_\alpha * s_{\beta/\lambda})[U] K_\lambda && \text{by (4.7)}, \\ &= \sum_\lambda U_{s_\alpha * s_{\beta/\lambda}} K_\lambda. \end{aligned}$$

□

4.5. **Operators U and $\overline{K}$.** We now proceed to proving (3.5):

$$\overline{K}_\beta U_\alpha = \sum_{\tau,\nu} U_{(s_{\beta/\nu} * s_\tau) s_{\alpha/\tau}} \overline{K}_\nu \,.$$

(The identity (3.6) is deduced by taking adjoints). Again, we first consider commutation relations for the generating series of the families of operators involved.

Lemma 4.5. *Let A and B be two alphabets. We have*

$$\overline{K}_{\sigma[BX]} U_{\sigma[AX]} = U_{\sigma[A(B+1)X]} \overline{K}_{\sigma[BX]}. \tag{4.17}$$

Proof. We have, for any symmetric function f,

$$\begin{aligned}
\overline{K}_{\sigma[BX]}U_{\sigma[AX]}(f) &= \overline{K}_{\sigma[BX]}(\sigma[AX]f[X]) \\
&= \sigma[-B]\sigma[AX(B+1)]f[X(B+1)] && \text{by (4.12),} \\
&= \sigma[AX(B+1)]\sigma[-B]f[X(B+1)] \\
&= \sigma[AX(B+1)]\overline{K}_{\sigma[BX]}(f) && \text{by (4.12) again,} \\
&= U_{\sigma[A(B+1)X]}\overline{K}_{\sigma[BX]}(f).
\end{aligned}$$

□

Proof of (3.5). From (4.17) we extract the term $\overline{K}_\beta U_\alpha$ by taking scalar product with $s_\alpha[A]\, s_\beta[B]$. This yields

$$\overline{K}_\beta U_\alpha = \left\langle U_{\sigma[A(B+1)X]}\overline{K}_{\sigma[BX]} \,\middle|\, s_\alpha[A]s_\beta[B] \right\rangle_{A,B}.$$

Expanding in the scalar product the generating function $\overline{K}_{\sigma[BX]}$, we get

$$\overline{K}_\beta U_\alpha = \sum_\nu \left\langle U_{\sigma[A(B+1)X]}s_\nu[B] \,\middle|\, s_\alpha[A]s_\beta[B] \right\rangle_{A,B} \overline{K}_\nu.$$

This simplifies as follows:

$$\begin{aligned}
\overline{K}_\beta U_\alpha &= \sum_\nu \left\langle \left\langle U_{\sigma[A(B+1)X]} \,\middle|\, s_{\alpha[A]} \right\rangle_A s_\nu[B] \,\middle|\, s_\beta[B] \right\rangle_B \overline{K}_\nu \\
&= \sum_\nu \left\langle \left\langle \sigma[A(B+1)U] \,\middle|\, s_{\alpha[A]} \right\rangle_A s_\nu[B] \,\middle|\, s_\beta[B] \right\rangle_B \overline{K}_\nu \\
&= \sum_\nu \left\langle s_{\alpha[(B+1)U]}s_\nu[B] \,\middle|\, s_\beta[B] \right\rangle_B \overline{K}_\nu && \text{by (4.8),} \\
&= \sum_\nu \left\langle s_{\alpha[BU+U]}s_\nu[B] \,\middle|\, s_\beta[B] \right\rangle_B \overline{K}_\nu \\
&= \sum_\nu \left\langle \sum_\tau s_\tau[BU]s_{\alpha/\tau}[U] \,\middle|\, s_{\beta/\nu}[B] \right\rangle_B \overline{K}_\nu \\
&= \sum_{\nu,\tau} \left\langle s_\tau[BU] \,\middle|\, s_{\beta/\nu}[B] \right\rangle_B s_{\alpha/\tau}[U]\overline{K}_\nu \\
&= \sum_{\nu,\tau} (s_\tau * s_{\beta/\nu})[U]s_{\alpha/\tau}[U]\overline{K}_\nu && \text{by (4.7),} \\
&= \sum_{\nu,\tau} U_{(s_\tau * s_{\beta/\nu})s_{\alpha/\tau}}\overline{K}_\nu.
\end{aligned}$$

□

In Section 8 we present as a application a combinatorial rule for the Kronecker product of any skew Schur function by $s_{(n-1,1)}$. Other interesting particular cases of (3.5) correspond to the cases when $\lambda = (k)$ (Kronecker product with a two-row shape) and $\lambda = (1^k)$ (Kronecker product with a hook), where we get:

$$\overline{K}_{(k)}U_\alpha = \sum_{j=0}^{k}\left(\sum_{\rho\vdash k-j} U_{\alpha/\rho}U_\rho\right)\overline{K}_{(j)}, \qquad \overline{K}_{(1^k)}U_\alpha = \sum_{j=0}^{k}\left(\sum_{\rho\vdash k-j} U_{\alpha/\rho}U_{\rho'}\right)\overline{K}_{(1^j)}.$$

Setting $n = |\alpha|$ and $m = |\gamma|$, the same identities can be stated as:

$$s_{(n+m-k,k)} * (s_\alpha s_\gamma) = \sum_{j=0}^{k} \left(\sum_{\rho \vdash k-j} s_{\alpha/\rho} s_\rho \right) (s_\gamma * s_{(m-j,j)}),$$

$$s_{(n+m-k,1^k)} * (s_\alpha s_\gamma) = \sum_{j=0}^{k} \left(\sum_{\rho \vdash k-j} s_{\alpha/\rho} s_{\rho'} \right) (s_\gamma * s_{(m-j,1^j)}).$$

The terms of the form $\sum_{\rho \vdash q} s_{\alpha/\rho} s_\rho$ and $\sum_{\rho \vdash q} s_{\alpha/\rho} s_{\rho'}$ that appear in these identities can take the following alternative forms, as a direct consequence of Littlewood's Identity (3.8):

$$\sum_{\rho \vdash q} s_{\alpha/\rho} s_\rho = s_\alpha * h_{(n-q,q)}, \qquad \text{and likewise } \sum_{\rho \vdash q} s_{\alpha/\rho} s_{\rho'} = s_\alpha * (h_{n-q} e_q).$$

5. Unicity

The identities in Theorem 3.1 and Corollary 3.2 express some operators as linear combinations of operators $U_\mu D_\nu$, $D_\nu U_\mu$, $U_\mu K_\nu$, etc. Are such expressions unique?

Lemma 5.1. *Let L and M be linear maps from Sym to* $\mathrm{End}(Sym)$. *Consider the linear map $\Phi_{L,M} : Sym \otimes_{\mathbb{Q}} Sym \to Sym$ that sends $f \otimes g$ to $L_f M_g$. Let (v_λ) and (w_λ) be bases of Sym.*

Any $\phi \in \mathrm{End}(Sym)$ has at most one expansion of the form

$$\sum_{\alpha,\beta} a_{\alpha,\beta} L_{v_\alpha} M_{w_\beta},$$

where the $a_{\alpha,\beta}$ are scalars, if and only if $\Phi_{L,M}$ is injective.

If $\Phi_{L,M}$ is injective, we will say that *expansions with respect to the ordered pair (L, M) are unique.* This amounts to saying that the operators $L_\lambda M_\mu$ are linearly independent. The lemma shows that this can be checked by proving that operators $L_{v_\lambda} M_{w_\mu}$ are linearly independent, for some good choice of bases (v_λ) and (w_λ).

Proposition 5.2. *Expansions with respect to the pairs (U, D), (D, U), (U, K), (K, D), $(U, \overline{K})$ and $(\overline{K}, D)$ are unique.*

Proof. It is well-known that the algebra generated by 1 and the operators U_{p_k} and D_{p_k} $(k > 0)$ is the bosonic creation and annihilation operator algebra (this appears for instance in [7]). This algebra is the algebra generated by 1 and operators a_n and $a_n^\dagger$, with $n > 0$, that commute except for

$$\left[a_n, a_n^\dagger\right] = 1, \qquad \text{for all } n > 0.$$

Equivalently, this is the Weyl algebra in infinitely many generators. The isomorphisms are given by:

$$U_{p_k} \mapsto a_k^\dagger \mapsto \widehat{x_k}\,,$$
$$D_{p_k} \mapsto k\, a_k \mapsto k\, \tfrac{\partial}{\partial x_k}\,.$$

It is also well-known that in the bosonic creation and annihilation operator algebra, the monomials in normal order

$$(a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2} \cdots a_1^{n_1} a_2^{n_2} \cdots$$

as well as the monomials in antinormal order

$$a_1^{n_1} a_2^{n_2} \cdots (a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2} \cdots$$

are linearly independent. This shows that the operators $U_{p_\lambda} D_{p_\mu}$ are linearly independent, and so are the operators $D_{p_\mu} U_{p_\lambda}$. And thus expansions with respect to (U, D) and expansions with respect to (D, U) are unique.

Let us consider expansions with respect to (U, K). For this we show that the operators $U_{p_\alpha} K_{p_\beta}$ are linearly independent. Let $F = \sum_{\alpha,\beta} a_{\alpha,\beta} U_{p_\alpha} K_{p_\beta}$ be a linear combination of these operators. Then, for any partition λ,

$$F(p_\lambda) = \sum_\alpha a_{\alpha,\lambda} p_\alpha z_\lambda p_\lambda.$$

This shows that, if $F = 0$, then $\sum_\alpha a_{\alpha,\lambda} p_\alpha = 0$ for all λ, and thus that $a_{\alpha,\lambda} = 0$ for all α and λ. This proves that the operators $U_{p_\alpha} K_{p_\beta}$ are linearly independent, and that expansions with respect to (U, K) are unique.

By taking adjoints, we get that expansions with respect to (K, D) are unique. Similarly, we prove that expansions with respect to $(U, \overline{K})$ are unique. For this we show that the operators $U_{p_\alpha} \overline{K}_{p_\beta[X+1]}$ are linearly independent. For any partition λ,

$$\begin{aligned} U_{p_\alpha} \overline{K}_{p_\beta[X+1]}(p_\lambda) &= p_\alpha \cdot (\Gamma_1 p_\beta[X+1] * p_\lambda) \\ &= p_\alpha \cdot ((\sigma[X] p_\beta[X]) * p_\lambda) \quad \text{by (4.13)} \\ &= p_\alpha \cdot \sum_\mu \frac{1}{z_\mu} ((p_\mu p_\beta) * p_\lambda) \\ &= \begin{cases} \frac{z_\lambda}{z_\mu} p_\alpha p_\lambda & \text{if there is a partition } \mu \text{ such that } \lambda = \mu \cup \beta, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Consider a linear relation $0 = \sum_{\alpha,\beta} a_{\alpha,\beta} U_{p_\alpha} \overline{K}_{p_\beta[X+1]}$. We get, by evaluating at p_λ,

$$0 = \sum_{\alpha,\ \beta \preceq \lambda} a_{\alpha,\beta} \frac{z_\lambda}{z_\mu} p_\alpha p_\lambda$$

where $\beta \preceq \lambda$ means that there exists a partition μ with $\beta \cup \mu = \lambda$. Therefore

$$0 = \sum_{\beta \preceq \lambda} a_{\alpha,\beta} \frac{z_\lambda}{z_\mu}$$

for all α and all λ. This can be rewritten as

$$0 = a_{\alpha,\lambda} + \text{ a linear combination of } a_{\alpha,\beta} \text{ for } \beta \preceq \lambda \text{ with } \beta \neq \lambda.$$

We deduce that all coefficients $a_{\alpha,\lambda}$ are zero.

By taking adjoint we get that expansions are unique for $(\overline{K}, D)$. □

By contrast, note that expansions with respect to (K, U) or to (D, K) are not unique. For instance we have the relation $K_{p_2} U_{p_1} = 0$, that is straightforwardly equivalent to the relation $K_2 U_1 = K_{1,1} U_1$.

6. The operators $\overline{K}_f$ are in the algebra generated by the operators U_g and D_h

Using Pieri's rule we easily see that (2.2) is equivalent to

$$\overline{K}_{(1)} = U_{(1)}D_{(1)} - 1\,. \tag{6.1}$$

This identity easily follows applying Littlewood's Identity (3.10) to the equation $s_{(n-1,1)} = s_{(n-1)}s_{(1)} - s_{(n)}$.

More generally we have the following result.

Proposition 6.1. *For any symmetric function f, the operator $\overline{K}_f$ is in the algebra generated by the operators U_g and D_h. Precisely:*

$$\overline{K}_f = \sum_\lambda U_{f[X-1]*s_\lambda} D_\lambda.$$

Proof. Littlewood's Identity (3.8) can be written as:

$$s_\beta * (s_\alpha s_\gamma) = \sum_\lambda \left(D_\lambda(s_\beta) * s_\alpha\right)\left(s_\lambda * s_\gamma\right)$$

for any three partitions α, β, γ. Extending this by linearity, we get that for any three symmetric functions f_1, f_2 and g,

$$g * (f_1 f_2) = \sum_\lambda \left(D_\lambda(g) * f_1\right)\left(s_\lambda * f_2\right) \tag{6.2}$$

Now, let f and g be two symmetric functions. We have

$$\begin{aligned}
\overline{K}_f(g) &= (\Gamma_1 f) * g = (\sigma[X]f[X-1]) * g \\
&= \sum_\lambda (\sigma[X] * D_\lambda(g))(f[X-1] * s_\lambda) && \text{by (6.2)} \\
&= \sum_\lambda D_\lambda(g)(f[X-1] * s_\lambda).
\end{aligned}$$

□

For instance, for $f = h_k$ we have: $\overline{K}_{(k)} = \sum_{\lambda\vdash k} U_\lambda D_\lambda - \sum_{\lambda\vdash k-1} U_\lambda D_\lambda$, since $h_k[X-1] = h_k - h_{k-1}$. With $k = 1$ we recover (6.1).

7. Application to the skew Littlewood–Richardson rule

In this section we present our first application of Theorem 3.1: a new proof of the skew Littlewood–Richardson rule as conjectured by Assaf and the second author [1] and proved by Lam, Lauve and Sottile [8]. As in [8], our starting point is (3.2). In [8], first an "algebraic skew Littlewood–Richardson rule" is derived, involving sums of products of Littlewood–Richardson coefficients. Then, the combinatorial skew Littlewood–Richardson rule is obtained by interpreting these Littlewood–Richardson coefficients as counting semistandard Young tableaux with given rectification. Our proof fits more closely to the statement of the skew Littlewood–Richardson rule: we avoid going through the algebraic skew Littlewood–Richardson rule and use the interpretation of the Littlewood–Richardson coefficients as counting semistandard Young tableaux with content and Yamanouchi constraints. Our proof appears in Subsections 7.2 and 7.3 and is largely combinatorial.

For a positive integer k and a partition γ, the classical Pieri rule [12] gives a simple and beautiful expression for the product $s_{(k)}s_\gamma$ as a sum of Schur functions.

A *k-horizontal* (resp. *k-vertical*) *strip* is a skew shape with k boxes that has at most one box in each column (resp. row). The Pieri rule states that

$$s_{(k)}s_\gamma = \sum_\lambda s_{\widehat{\gamma}} \,,$$

where the sum is over all partitions $\widehat{\gamma}$ such that $\widehat{\gamma}/\gamma$ is a k-horizontal strip. In [1], Assaf and the second author generalized the Pieri rule to the setting of skew shapes as follows:

$$s_{(k)}s_{\gamma/\beta} = \sum_{i=0}^{k}(-1)^i \sum_{\widehat{\gamma},\check{\beta}} s_{\widehat{\gamma}/\check{\beta}} \,, \tag{7.1}$$

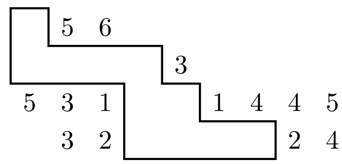
where the sum is over all partitions $\widehat{\gamma}$ and $\check{\beta}$ such that $\widehat{\gamma}/\gamma$ is a $(k-i)$-horizontal strip, and $\beta/\check{\beta}$ is an i-vertical strip. We will use the skew Pieri rule with $k=1$ in Section 8.

For the next level of generality, it is natural to ask for a similarly combinatorial expression for $s_\alpha s_{\beta/\gamma}$ for any partition α. Equation (3.11) gives one expression, but it does not mimic (7.1) in the sense that it does not give the answer as a signed sum of skew Schur functions. Instead, the skew Littlewood–Richardson rule [8] gives an expression for the even more general product $s_{\alpha/\delta}s_{\beta/\gamma}$ as a signed sum of skew Schur functions. In this section we will derive the skew Littlewood–Richardson rule from (3.2) in the following way. In Subsection 7.2, we will use a combinatorial approach to obtain from (3.2) the skew Littlewood–Richardson rule in the case when δ is empty, and then we will use a linearity argument to derive the result for general δ in Subsection 7.3.

7.1. **The combinatorial skew Littlewood–Richardson rule.** In order to state the skew Littlewood–Richardson rule, we first need some terminology. As usual, a sequence of positive integers ω is said to be a *lattice permutation* if any prefix of ω contains at least as many appearances of i as $i+1$, for all $i \geqslant 1$. For a partition δ, we will say that ω is a δ-lattice permutation if the word obtained by prefixing ω with δ_1 copies of 1 followed by δ_2 copies of 2, etc., is a lattice permutation.

We will draw our Young tableaux in French notation, implying that the entries of an SSYT weakly increase along the rows and strictly increase up the columns. An *anti-semistandard Young tableau* (ASSYT) T_1 of shape α/β is a filling of the boxes of α/β so that the entries strictly decrease along the rows and weakly decrease up the columns. Equivalently, T_1 is an ASSYT if the tableau $(T_1')^r$ obtained by transposing T_1 and then rotating it 180° is an SSYT. The *reverse reading word of an SSYT T_2* is defined as usual as the word obtained by reading right-to-left along the rows of T_2, taking the rows from bottom to top. In contrast, the *reverse reading word of an ASSYT T_1* is the word obtained by reading up the columns of T, taking the columns from right-to-left. Equivalently, we can take the usual reverse reading work of the SSYT $(T_1')^r$. Given a pair of tableaux (T_1, T_2), where T_1 is an ASSYT and T_2 is an SSYT, we define the *reverse reading word of the pair* as the concatenation of the reading word of T_1 with that of T_2. We will encounter such pairs as in the the figure below, where the entries in the bottom left form an ASSYT, and the entries above or to the right of the outlined skew shape form an

SSYT.



(7.2)

The reverse reading word of (T_1, T_2) shown in (7.2) is 21335425441365, which is certainly not a lattice permutation but is a 5321-lattice permutation.

We are now ready to state the skew Littlewood–Richardson rule.

Theorem 7.1 (Conjecture 6.1 of [1]; Theorem 3.2 and Remark 3.3(ii) of [8])**.** *For skew shapes α/δ and γ/β,*

$$s_{\alpha/\delta} s_{\gamma/\beta} = \sum_{\substack{T_1 \in \mathrm{ASSYT}(\beta/\breve{\beta}) \\ T_2 \in \mathrm{SSYT}(\widehat{\gamma}/\gamma)}} (-1)^{|\beta/\breve{\beta}|} s_{\widehat{\gamma}/\breve{\beta}}\ , \tag{7.3}$$

where the sum is over all ASSYT T_1 of shape $\beta/\breve{\beta}$ for some $\breve{\beta} \subseteq \beta$, and SSYT T_2 of shape $\widehat{\gamma}/\gamma$ for some $\widehat{\gamma} \supseteq \gamma$, with the following properties:

(a) *the combined content of T_1 and T_2 is the component-wise difference $\alpha - \delta$, and*
(b) *the reverse reading word of (T_1, T_2) is a δ-lattice permutation.*

For example, the ASSYT and SSYT pair of (7.2) contribute $-s_{9953/1}$ to the product $s_{755431/5321} s_{7541/33}$. Note that when β and δ are empty, we recover the classical Littlewood–Richardson rule.

7.2. **Recovering a special case of the combinatorial skew Littlewood–Richardson rule.** Our first step to reproving Theorem 7.1 is to start with (3.2) and show it implies Theorem 7.1 in the case when $\delta = (0)$, the empty partition. Instead of (3.2), we work with the equivalent (3.11):

$$s_\alpha s_{\gamma/\beta} = \sum_\lambda (-1)^{|\lambda|} D_{\beta/\lambda'}(s_{\alpha/\lambda} s_\gamma).$$

First, let us examine the product $s_{\alpha/\lambda} s_\gamma$ from the right–hand side, and expand it in terms of Schur functions. Note that only those s_ν with $\nu \supseteq \gamma$ will appear in the Schur expansion with nonzero coefficient. Thus we can write

$$s_{\alpha/\lambda} s_\gamma = \sum_{\widehat{\gamma} \supseteq \gamma} a_{\widehat{\gamma}} s_{\widehat{\gamma}}\ .$$

We have

$$a_{\widehat{\gamma}} = \left\langle\, s_{\widehat{\gamma}} \,\middle|\, s_{\alpha/\lambda} s_\gamma \,\right\rangle = \left\langle\, s_{\widehat{\gamma}/\gamma} \,\middle|\, s_{\alpha/\lambda} \,\right\rangle = \left\langle\, s_{\widehat{\gamma}/\gamma} s_\lambda \,\middle|\, s_\alpha \,\right\rangle.$$

The product $s_{\widehat{\gamma}/\gamma} s_\lambda$ is equal to the skew Schur function of the shape $(\widehat{\gamma}/\gamma) \oplus \lambda$. (The notation $\oplus$ denotes that $\widehat{\gamma}/\gamma$ is positioned so that its bottom-right corner box is immediately northwest of the top-left corner box of λ.) Therefore, the coefficient $a_{\widehat{\gamma}}$ is equal to the number of Littlewood–Richardson fillings (LR-fillings) of that skew shape that have content α. Any LR-filling of that shape must just fill the ith row of λ with the number i, for all i. Thus $a_{\widehat{\gamma}}$ equals the number of SSYT of shape

$\widehat{\gamma}/\gamma$ whose reverse reading word is a λ-lattice permutation and whose content is the component-wise difference $\alpha - \lambda$. Hence (3.11) is equivalent to

$$s_\alpha s_{\gamma/\beta} = \sum_\lambda (-1)^{|\lambda|} D_{\beta/\lambda'} \sum_{T_2} s_{\widehat{\gamma}} \tag{7.4}$$

where second sum is over all SSYT T_2 of shape $\widehat{\gamma}/\gamma$ for some $\widehat{\gamma} \supseteq \gamma$, and content $\alpha - \lambda$, whose reverse reading word is a λ-lattice permutation.

Next, we will examine the term $s_{\beta/\lambda'}$. The coefficient of s_ν in this term is exactly the Littlewood–Richardson coefficient $c^{\beta}_{\lambda'\nu}$, which is nonzero only if $\nu \subseteq \beta$. Thus we wish to determine the coefficient of $s_{\check{\beta}}$ in $s_{\beta/\lambda'}$ when $\check{\beta} \subseteq \beta$, which equals the number of LR-fillings of $\beta/\check{\beta}$ of content λ'. We claim that such fillings T are in a shape-preserving bijection with ASSYT that are lattice permutations of content λ (as opposed to content λ' previously). Indeed the bijection ψ is defined as mapping the ith appearance (in the reverse reading word of the SSYT T) of the number j to the number i, for all i and j. For example,

$$\begin{array}{cccccc} 3 & 4 & & & & \\ 2 & 3 & 3 & & & \\ & 1 & 2 & 2 & & \\ & & 1 & 1 & 1 & 1 \end{array} \quad \longrightarrow \quad \begin{array}{cccccc} 3 & 1 & & & & \\ 3 & 2 & 1 & & & \\ & 5 & 2 & 1 & & \\ & & 4 & 3 & 2 & 1 \end{array} .$$

Then, one can check that ψ has the following necessary properties.

- ◦ The inverse of ψ is given by the ASSYT analogue of ψ: map the jth appearance (in the reverse reading word, now in the ASSYT sense) of the number i to the number j, for all i and j.
- ◦ The image $\psi(T)$ of an LR-filling T is indeed an ASSYT whose reverse reading word is a lattice permutation.
- ◦ Such a $\psi(T)$ maps to an LR-filling under the inverse map.
- ◦ Both ψ and its inverse transpose the content partition.

Thus (7.4) is equivalent to

$$s_\alpha s_{\gamma/\beta} = \sum_\lambda (-1)^{|\lambda|} \sum_{T_1} D_{\check{\beta}} \sum_{T_2} s_{\widehat{\gamma}} = \sum_\lambda (-1)^{|\lambda|} \sum_{T_1, T_2} s_{\widehat{\gamma}/\check{\beta}},$$

where the relevant sums are over all T_1 and T_2 such that

- ◦ T_1 is an ASSYT having content λ, whose reverse reading word is a lattice permutation, and with shape $\beta/\check{\beta}$ for some $\check{\beta} \subseteq \beta$, and
- ◦ T_2 is an SSYT having content $\alpha - \lambda$, a λ-lattice permutation as reverse reading word, and shape $\widehat{\gamma}/\gamma$ for some $\widehat{\gamma} \supseteq \gamma$.

Note that T_1 tells us that $|\lambda| = |\beta/\check{\beta}|$, and we have arrived at Theorem 7.1 in the case when $\delta = (0)$.

7.3. **Recovering the full combinatorial skew Littlewood–Richardson rule.** Our second step is to use a linearity argument to derive Theorem 7.1 for general δ. For this, observe that the coefficient of $(-1)^{|\beta/\check{\beta}|} s_{\widehat{\gamma}/\check{\beta}}$ on the right-hand side of (7.3) is the number of pairs of tableaux (T_1, T_2) with T_1 an ASSYT and T_2 a SSYT, fulfilling conditions (a) and (b) in Theorem 7.1. But T_1 being an ASSYT is equivalent to $(T_1')^r$ being an SSYT, and the reverse reading word of the ASSYT T_1 is defined so that $(T_1')^r$ has the same reverse reading word as an SSYT. Therefore, the coefficient of $(-1)^{|\beta/\check{\beta}|} s_{\widehat{\gamma}/\check{\beta}}$ on the right-hand side of (7.3) equals the number of SSYT of shape $(\widehat{\gamma}/\gamma) \oplus (\beta'/\breve{\beta}')^r$ and content α whose reverse reading word is a δ–lattice permutation. This is the number of SSYT of shape $(\widehat{\gamma}/\gamma) \oplus (\beta'/\breve{\beta}')^r \oplus \delta$ and content α whose reverse reading word is a lattice permutation. By the Littlewood–Richardson rule, this quantity equals the coefficient of s_α in the Schur expansion of $s_{(\widehat{\gamma}/\gamma)\oplus(\beta'/\breve{\beta}')^r\oplus\delta}$. This skew Schur function being equal to the product $s_{\widehat{\gamma}/\gamma} s_{(\beta'/\breve{\beta}')^r} s_\delta$, this coefficient is equal to

$$\left\langle s_{\widehat{\gamma}/\gamma} s_{(\beta'/\breve{\beta}')^r} s_\delta \,\middle|\, s_\alpha \right\rangle ,$$

which is equal to

$$\left\langle s_{\widehat{\gamma}/\gamma} s_{(\beta'/\breve{\beta}')^r} \,\middle|\, s_{\alpha/\delta} \right\rangle .$$

Therefore, (7.3) is equivalent to

$$s_{\alpha/\delta} s_{\gamma/\beta} = \sum_{\widehat{\gamma},\breve{\beta}'} (-1)^{|\beta/\breve{\beta}|} \left\langle s_{\widehat{\gamma}/\gamma} s_{(\beta'/\breve{\beta}')^r} \,\middle|\, s_{\alpha/\delta} \right\rangle s_{\widehat{\gamma}/\breve{\beta}} , \tag{7.5}$$

where the sums are over all partitions $\widehat{\gamma}$ and $\breve{\beta}'$ such that $\widehat{\gamma} \supseteq \gamma$ and $\breve{\beta} \subseteq \beta$.

The key observation is that (7.5) is linear in $s_{\alpha/\delta}$. Given any partitions β and γ, (7.5) will be true for all partitions α and δ when

$$f \cdot s_{\gamma/\beta} = \sum_{\widehat{\gamma},\breve{\beta}'} (-1)^{|\beta/\breve{\beta}|} \left\langle s_{\widehat{\gamma}/\gamma} s_{(\beta'/\breve{\beta}')^r} \,\middle|\, f \right\rangle s_{\widehat{\gamma}/\breve{\beta}} ,$$

holds for any symmetric function f. Since the Schur functions form a basis for the space of symmetric functions, it is enough to check it for $f = s_\alpha$, all partitions α. This is what was done in 7.2.

8. A combinatorial interpretation for the Kronecker product of a skew Schur function by $s_{(n-1,1)}$.

As another application of the identities of Section 3, our goal for this section is to derive a combinatorial formula for Kronecker products involving skew Schur functions. Let α be a partition of n, and let us speak of partitions and their Young diagrams interchangeably.

We aim to generalize (2.2) to skew Schur functions. This leads to our next use of the identities of Section 2. Corollary 3.3 implies the relation $[D_\theta, \overline{K}_{(1)}] = D_{s_{\theta/(1)}s_1}$, which gives

$$\overline{K}_{(1)} D_\theta(s_\alpha) = D_\theta \overline{K}_{(1)}(s_\alpha) - D_{s_{\theta/(1)}s_1}(s_\alpha).$$

Applying (2.2) and the fact that

$$s_{\theta/(1)} s_1 = \#\text{corners}(\theta) s_\theta + \sum_{\phi \in \theta^{\mp}} s_\phi ,$$

we get

$$s_{\alpha/\theta} * s_{(n-|\theta|-1,1)} = (\#\text{corners}(\alpha) - \#\text{corners}(\theta) - 1) s_{\alpha/\theta} + \sum_{\beta \in \alpha^{\mp}} s_{\beta/\theta} - \sum_{\phi \in \theta^{\mp}} s_{\alpha/\phi} ,$$

Thus we have an algebraic proof of the following.

Theorem 8.1. *Suppose $\alpha \vdash n$ and $\theta \vdash k$ with $\theta \subseteq \alpha$. Then*

$$s_{\alpha/\theta} * s_{(n-k-1,1)} = (\#\text{corners}(\alpha) - \#\text{corners}(\theta) - 1) s_{\alpha/\theta} + \sum_{\beta \in \alpha^{\mp}} s_{\beta/\theta} - \sum_{\phi \in \theta^{\mp}} s_{\alpha/\phi} .$$

This results begs for a a combinatorial proof. We offer a proof which is "two-thirds" combinatorial. The part which is non-combinatorial makes use of (6.1), which is turn is proved using Littlewood's Identity (3.10).

Proof 2 of Theorem 8.1. The proof will work in three stages. In the short first stage, which is the non-combinatorial one, we will apply (6.1) to express $s_{\alpha/\theta} * s_{(n-k-1,1)}$ in a form (8.1) not involving any Kronecker products. Then, using the skew Pieri rule (7.1), we will reduce the problem to showing an identity (8.2) that is effectively purely about SSYT. This identity will be proved in the third stage using jeu de taquin.

First, we will need to determine the result of applying $D_{(1)}$ to the skew Schur function $s_{\alpha/\theta}$. We have

$$\left\langle s_\beta \,\middle|\, D_{(1)} s_{\alpha/\theta} \right\rangle = \left\langle s_\beta s_{(1)} \,\middle|\, s_{\alpha/\theta} \right\rangle = \sum_{\delta \in \theta^+} \left\langle s_\beta s_\delta \,\middle|\, s_\alpha \right\rangle = \left\langle s_\beta \,\middle|\, \sum_{\delta\in\theta^+} s_{\alpha/\delta} \right\rangle,$$

and so $D_{(1)} s_{\alpha/\theta} = \sum_{\delta\in\theta^+} s_{\alpha/\delta}$. Applying (6.1) to $s_{\alpha/\theta}$, we immediately deduce

$$s_{\alpha/\theta} * s_{(n-k-1,1)} = s_{(1)} \sum_{\delta\in\theta^+} s_{\alpha/\delta} - s_{\alpha/\theta}\,. \tag{8.1}$$

We next wish to apply the skew Pieri rule to $s_{(1)} \sum_{\delta\in\theta^+} s_{\alpha/\delta}$, but it will prove worthwhile to perform a preliminary step. By definition, $s_{\alpha/\delta} = 0$ unless $\delta \subseteq \alpha$, so it suffices to sum over those δ such that $\delta \in \theta^+$ and $\delta \subseteq \alpha$. We will denote that δ satisfies both conditions by writing $\delta \in \theta^{+\alpha}$. So we now apply the skew Pieri rule to

$$s_{\alpha/\theta} * s_{(n-k-1,1)} = s_{(1)} \sum_{\delta\in\theta^{+\alpha}} s_{\alpha/\delta} - s_{\alpha/\theta}$$

to yield

$$s_{\alpha/\theta} * s_{(n-k-1,1)} = \sum_{\substack{\gamma\in\alpha^+\\ \delta\in\theta^{+\alpha}}} s_{\gamma/\delta} - \sum_{\substack{\delta\in\theta^{+\alpha}\\ \phi\in\delta^-}} s_{\alpha/\phi} - s_{\alpha/\theta}\ .$$

Let us examine the second sum. For any δ in $\theta^{+\alpha}$, we can choose $\phi = \theta$. We see that the other ϕ that arise will be exactly those elements of $\theta^{\mp}$ that are contained in α. Therefore,

$$s_{\alpha/\theta} * s_{(n-k-1,1)} = \sum_{\substack{\gamma\in\alpha^+\\ \delta\in\theta^{+\alpha}}} s_{\gamma/\delta} - |\theta^{+\alpha}| s_{\alpha/\theta} - \sum_{\phi\in\theta^{\mp}} s_{\alpha/\phi} - s_{\alpha/\theta}\ .$$

Thus to prove Theorem 8.1, it remains to show that

$$\sum_{\substack{\gamma\in\alpha^+\\ \delta\in\theta^{+\alpha}}} s_{\gamma/\delta} - |\theta^{+\alpha}| s_{\alpha/\theta} = (\#\text{corners}(\alpha) - \#\text{corners}(\theta)) s_{\alpha/\theta} + \sum_{\beta\in\alpha^{\mp}} s_{\beta/\theta}\ . \tag{8.2}$$

Our main tool for proving the above identity will be jeu de taquin but, like with our application of the skew Pieri rule, it will be worthwhile to rewrite (8.2) in a slightly different form. Observe that for any partition α, we have $\#\text{corners}(\alpha) = |\alpha^+| - 1$. For $\theta \subseteq \alpha$, denote those elements of θ^+ that are not contained in α by $\theta^{+\alpha^c}$, which we can check can only be non-empty if α/θ has some empty rows or columns. We can now rewrite (8.2) as

$$\sum_{\substack{\gamma\in\alpha^+\\ \delta\in\theta^{+\alpha}}} s_{\gamma/\delta} = (|\alpha^+| - |\theta^{+\alpha^c}|) s_{\alpha/\theta} + \sum_{\beta\in\alpha^{\mp}} s_{\beta/\theta}\ . \tag{8.3}$$

For intuition, we can call the positions of the form λ/α for some $\lambda \in \alpha^+$ the *outside corners* of α. Then the term $|\alpha^+| - |\theta^{+\alpha^c}|$ is the number of outside corners of α, excluding those that are also outside corners of θ. See Example 8.2 below for a fully worked example of the remainder of the proof.

To prove (8.3) using jeu de taquin (jdt), consider an SSYT T that contributes to the left-hand side, meaning T has shape γ/δ, where $\gamma \in \alpha^+$ and $\delta \in \theta^{+\alpha}$. Notice that the unique box b of δ/θ is not an element of T, and that the unique box c of γ/α is in T. Perform a jdt slide of T into b, and let T' denote the resulting SSYT. There are three possibilities that can arise.

(a) $T' = T$, meaning there is no way to fill b under a jdt slide of T.
(b) T' contains b, and the single vacated box under the jdt slide is not c.
(c) T' contains b, and the single vacated box under the jdt slide is c.

By definition of jdt, Case (a) can happen if and only if b is a corner of γ. Since $b \in \delta \subseteq \alpha \subset \gamma$, it must also be the case that b is a corner of α. Therefore, since b is the unique box of δ/θ and is not an element of T' while c is the unique box of γ/α and is in T', the shape γ/δ of T' can be written in the equivalent form β/θ, where β is obtained from α by removing b and adding c. In particular, $\beta \in \alpha^{\mp}$, and so such T' contribute part of the sum on the right of (8.3).

We claim that Case (b) contributes the rest of the sum on the right of (8.3). We see that the shape of T' is obtained from the shape of T by making exactly two changes: T' contains b, and a box c' different from c has been vacated. As a result, T' has shape β/θ for some $\beta \in \alpha^{\mp}$.

To prove our claim from the start of the previous paragraph, let T' be an SSYT of shape β/θ, where $\beta \in \alpha^{\mp}$. We wish to show that T' arises as the image under jdt of exactly one T from Cases (a) and (b). In short, the reason is that jdt slides are reversible, but let us be more precise. Suppose β is obtained from α by removing a box d and adding a different box e. Perform a jdt slide of T' into d. (Some references would call this a *reverse* jdt slide, since d is outside β.) There are two possibilities that can arise.

(i) There is no way to fill d under a jdt slide of T'. This can happen if and only if d is an outside corner of θ. Thus β/θ can be written in the equivalent form γ/δ, where γ equals α with e added, and δ equals θ with d added. Since d is a corner of α, we also have that d is a corner of γ. These are exactly the conditions for T' to arise as an image under jdt of a T from Case (a) (where in fact $T = T'$ and our d here corresponds to b in Case (a)).
(ii) Performing a jdt slide of T' into d fills d and vacates an outside corner of θ. This is exactly the reverse of the jdt slide from Case (b) (where d is playing the role of c').

Thus T' arises as the image of a single T from Cases (a) and (b).

It remains to consider Case (c). Note that all T' in Case (c) are of shape α/θ. We would like to show that each T' is the image under jdt of k distinct T, where $k = (|\alpha^+| - |\theta^{+\alpha^c}|)$. This would show that the T' from Case (c) together contribute the term $(|\alpha^+| - |\theta^{+\alpha^c}|)s_{\alpha/\theta}$ from the right-hand side of (8.3), and (8.3) would be proved.

So pick a T' from Case (c). Pick an outside corner c of α, and perform a (reverse) jdt slide of T' into c. If c is not an outside corner of θ, then this jdt slide will fill

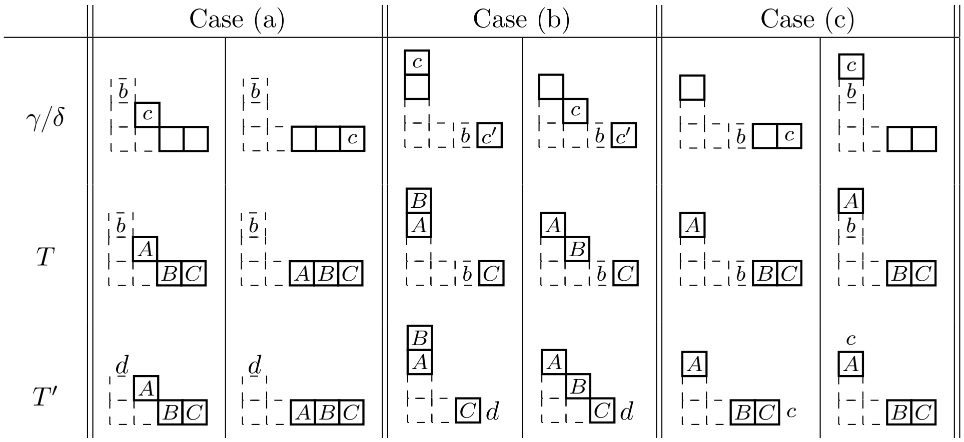
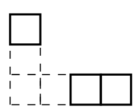


TABLE 8.1. The full set of skew shapes and SSYT for Example 8.2. In the first row, the dashed boxes are those in δ, and the solid boxes are those in γ/δ. Lowercase letters correspond to notation in the proof of (8.3), while the uppercase A, B and C denote entries of the SSYT that we assume satisfy the necessary inequalities but are otherwise any positive integers.

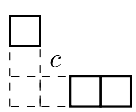
c, and the result will be an SSYT T of shape γ/δ with $\gamma \in \alpha^+$ and $\delta \in \theta^{+\alpha}$. These are exactly the conditions for T' to arise in Case (c).

On the other hand, if c is an outside corner of α and also of θ, then T' will remain fixed under the (reverse) jdt slide. Thus any T that maps to such a T' under a jdt slide must also have shape α/θ. Such a T from the left-hand side of (8.3) does not exist, since T would contain the single box γ/α whereas T' does not. We conclude that each T' in Case (c) is the image of exactly k distinct T, as required. □

Example 8.2. Suppose $\alpha = (4, 1, 1)$ and $\theta = (2, 1)$.

$\alpha/\theta =$ 

The complete set of shapes γ/δ, and SSYT T and T' from the proof of (8.3) are show in Table 8.1. Deliberately omitted from the table is the scenario from the last paragraph of the proof, where c is an outside corner of both α and θ, which does not contribute to either side of (8.3). In this example, there is $1 = |\theta^{+\alpha^c}|$ such situation, shown below.

$\alpha/\theta =$ 

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