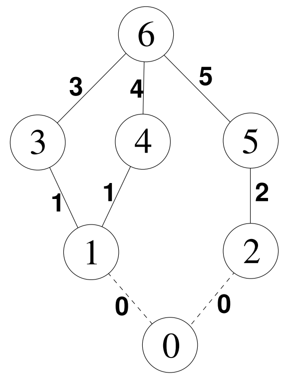
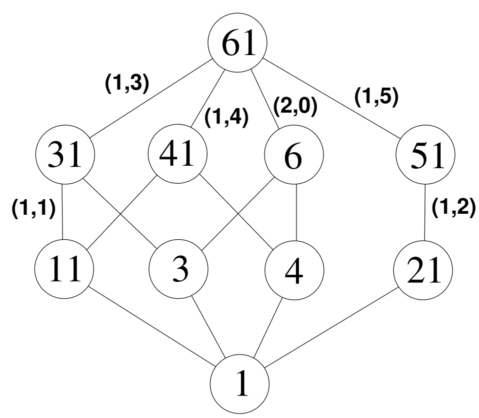
Discrete Morse theory for posets: the bare bones
with an example from generalized subword order

P_0 :



$$6 \xrightarrow{3} 3 \xrightarrow{1} 1 \xrightarrow{0} 0$$
$$6 \xrightarrow{4} \boxed{4} \xrightarrow{1} 1 \xrightarrow{0} 0$$
$$6 \xrightarrow{5} \boxed{5 \xrightarrow{2} 2} \xrightarrow{0} 0 \quad \Rightarrow \mu_0(0,6) = (-1)^{1-1} = +1$$

P^*:



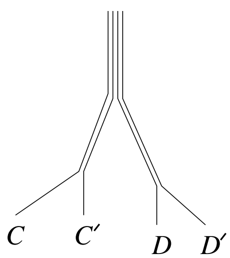
$$61 \xrightarrow{(1,3)} 31 \xrightarrow{(1,1)} 11 \xrightarrow{(1,0)} 01$$
$$61 \xrightarrow{(1,3)} 31 \xrightarrow{(2,0)} 30 \xrightarrow{(1,1)} 10$$
$$61 \xrightarrow{(1,4)} 41 \xrightarrow{(1,1)} 11 \xrightarrow{(1,0)} 01$$
$$61 \xrightarrow{(1,4)} 41 \xrightarrow{(2,0)} 40 \xrightarrow{(1,1)} 10$$
$$61 \xrightarrow{(1,5)} 51 \xrightarrow{(1,2)} 21 \xrightarrow{(1,0)} 01$$
$$61 \xrightarrow{(2,0)} 60 \xrightarrow{(1,3)} 30 \xrightarrow{(1,1)} 10$$
$$61 \xrightarrow{(2,0)} 60 \xrightarrow{(1,4)} 40 \xrightarrow{(1,1)} 10$$

Goal: compute $\mu(x, y)$ using DMT.

(1) Pick an ordering, denoted $\prec$, of the maximal chains of $[x, y]$ that is a *poset lexicographic order* (PLO). *Note:* chains are read from top to bottom.
(2) Identify the *skipped intervals (SIs)* of each maximal chain C, i.e., an interval I of C such that $C \setminus I \subseteq B$ for some maximal chain $B \prec C$.
(3) Identify the *minimal skipped intervals (MSIs)* of C, i.e., the SIs that are minimal with respect to containment.
(4) *Remove overlaps* among the MSIs of C in a certain precise fashion to obtain the set $\mathcal{J}(C)$ of intervals.
(5) If the $\mathcal{J}(C)$ cover the interior of C, then C is *critical.*
(6) *Compute* the Möbius function:

$$\mu(x, y) = \sum_{\text{critical chains } C} (-1)^{|\mathcal{J}(C)|-1}.$$

"Definition". An ordering $C_1 \prec C_2 \prec \cdots$ of the maximal chains of $[x, y]$ is a *poset lexicographic order (PLO)* if it satisfies the following (mild) property. Suppose that C and C' diverge from D and D' at a certain point, while the divergence of C' from C and of D' and D happens later. In this situation we insist that $C \prec D$ if and only if $C' \prec D'$.



Example. The construction of the disjoint sets $\mathcal{J}(C)$ from the MSIs of a maximal chain C is an iterative procedure illustrated in the table below. Suppose C is the maximal chain

$$y > c_1 > c_2 > \cdots > c_8 > x$$

and the MSIs from top to bottom are as given in the first line of the table.

MSIs	$I_1 = \{c_1, c_2\}$	$I_2 = \{c_2, c_3, c_4\}$	$I_3 = \{c_4, c_5, c_6\}$	$I_4 = \{c_5, c_6, c_7, c_8\}$
I_1 disjoint	$\boldsymbol{J_1} = \{c_1, c_2\}$	$I_2' = \{c_3, c_4\}$	$I_3' = \{c_4, c_5, c_6\}$	$I_4' = \{c_5, c_6, c_7, c_8\}$
I_2' disjoint	$J_1 = \{c_1, c_2\}$	$\boldsymbol{J_2} = \{c_3, c_4\}$	$I_3'' = \{c_5, c_6\}$	$I_4'' = \{c_5, c_6, c_7, c_8\}$
$\mathcal{J}(C)$	$J_1 = \{c_1, c_2\}$	$J_2 = \{c_3, c_4\}$	$\boldsymbol{J_3} = \{c_5, c_6\}$	no longer minimal

Note that the $\mathcal{J}(C)$ no longer cover the interior of C, so C is not a critical chain.

References

[1] Eric Babson and Patricia Hersh. Discrete Morse functions from lexicographic orders. *Trans. Amer. Math. Soc.*, 357(2):509–534 (electronic), 2005.

[2] Peter R. W. McNamara and E. Sagan, Bruce. The Möbius function of generalized subword order. *Adv. Math.*, 229(5):2741–2766, 2012.

[3] Bruce E. Sagan and Vincent Vatter. The Möbius function of a composition poset. *J. Algebraic Combin.*, 24(2):117–136, 2006.