When do quasisymmetric functions know that trees are different?

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Joint work with:
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UBC Discrete Mathematics Seminar
31 January 2023

Slides and paper available from
`http://www.unix.bucknell.edu/~pm040/`

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Outline

- Chromatic (quasi)symmetric functions and the motivating conjectures
- Converting to a poset question; more conjectures
- Some old and new results

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- More conjectures

The chromatic polynomial

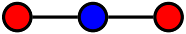
George Birkhoff, 1912

Graph $G = (V, E)$

Colouring/Coloring: a map $\kappa : V \to \{1, 2, 3, \ldots\}$

Proper coloring: adjacent vertices get different colors.

Proper



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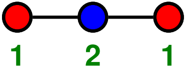

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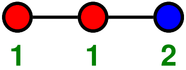

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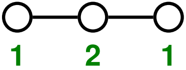

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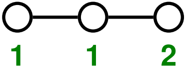

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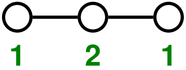

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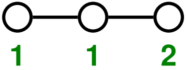


Chromatic polynomial: $\chi_G(k)$ is the number of proper colorings of G when k colors are available.

Example. $\chi_G(k) = k(k-1)(k-1)$

The chromatic symmetric function

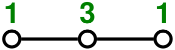
Richard Stanley, 1995

Graph $G = (V, E)$

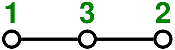
$V = \{v_1, v_2, \ldots, v_n\}$

To a proper coloring κ, we associate the monomial in commuting variables $x_1, x_2, \ldots$

$$x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}.$$



$x_1^2 x_3$



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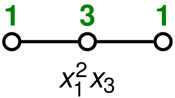

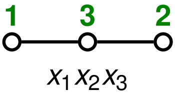

Chromatic symmetric function:

$$X_G(x_1, x_2, \ldots) = X_G(\mathbf{x}) = \sum_{\text{proper } \kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}.$$

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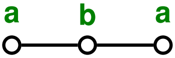

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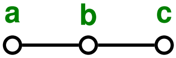

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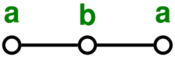

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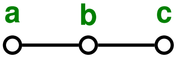

$x_a x_b x_c$

$$X_G(\mathbf{x}) = \sum_{a \neq b} x_a^2 x_b + 6 \sum_{a<b<c} x_a x_b x_c$$

$$(= m_{21} + 6m_{111}).$$

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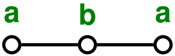

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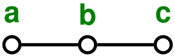

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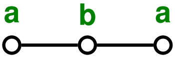

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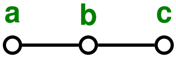

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- $X_G(\mathbf{x})$ is a symmetric function (invariant when you permute the colors/variables)
- Setting $x_i = 1$ for $1 \leq i \leq k$ and $x_i = 0$ otherwise yields $\chi_G(k)$.
  e.g. $k(k-1) + 6\binom{k}{3} = k(k-1)^2$.

Can $X_G(\mathbf{x})$ distinguish graphs?

$$X_G(\mathbf{x}) = \sum_{\text{proper } \kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}.$$

Statement 1.

$X_G(\mathbf{x})$ distinguishes graphs.

In other words, if G and H are not isomorphic, then $X_G(\mathbf{x}) \neq X_H(\mathbf{x})$.

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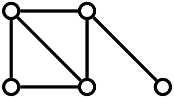

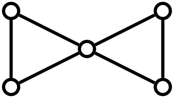

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if T and U are non-isomorphic trees, then $X_T(\mathbf{x}) \neq X_U(\mathbf{x})$.

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[Aliniaeifard, Aliste-Prieto, Crew, de Mier, Fougere, Heil, Ji, Loebl, Martin, Morin, Orellana, Scott, Smith, Sereni, Spirkl, Tian, Wagner, Wang, van Willigenburg, Zamora, ...]

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Remark. Stanley–Stembridge: another famous $X_G(\mathbf{x})$ conjecture.

A little bit of (quasi)symmetric functions

$x^2y + y^2x + x^2z + z^2x + y^2z + z^2y$ is a symmetric polynomial in $\{x, y, z\}$ because it doesn't change when you permute the variables.

$\sum_{a \neq b} x_a^2 x_b = x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + \cdots$ is a symmetric function in $\mathbf{x}$.

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Definition. A quasisymmetric function is a formal power series (over $\mathbb{Z}$, say) in $x_1, x_2, \ldots$ of bounded degree whose coefficients are *shift invariant* meaning

$$\text{coefficient of } x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k} = \text{coefficient of } x_{a_1}^{\alpha_1} x_{a_2}^{\alpha_2} \cdots x_{a_k}^{\alpha_k}$$

whenever $a_1 < a_2 < \cdots < a_k$.

$$M_{12} = \sum_{a<b} x_a x_b^2 = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + x_1 x_4^2 + \cdots .$$

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For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ the monomial quasisymmetric function is:

$$M_\alpha = \sum_{a_1<a_2<\cdots<a_k} x_{a_1}^{\alpha_1} x_{a_2}^{\alpha_2} \cdots x_{a_k}^{\alpha_k}.$$

The M_α form a basis for the algebra *QSym* of quasisymmetric functions.

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A great basis: Gessel's fundamental quasisymmetric functions

$$F_\alpha = \sum_{\beta \text{ refines } \alpha} M_\beta.$$

Example.

$$F_{32} = M_{32} + M_{212} + M_{122} + M_{1112} + M_{311} + M_{2111} + M_{1211} + M_{11111}.$$

(M_{221}, for example, does not appear).

The chromatic quasisymmetric function

John Shareshian & Michelle Wachs, 2014; Brittney Ellzey, 2017.

Directed graph $\overrightarrow{G} = (V, E)$.

Ascent of proper coloring κ: directed edge $u \to v$ with $\kappa(u) < \kappa(v)$

asc(κ): the number of ascents of κ.

Example. Colors $a < b < c$

$\kappa(v_1)$	$\kappa(v_2)$	$\kappa(v_3)$	asc(κ)
a	b	c	1
a	c	b	2
b	a	c	0
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c	a	b	0
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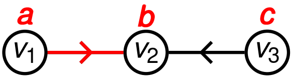

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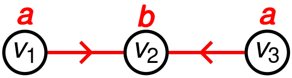

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Example. $X_{\overrightarrow{G}}(\mathbf{x}, t) = (2 + 2t + 2t^2) M_{111} + t^2 M_{21} + M_{12}.$

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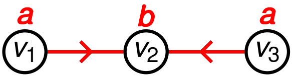

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b	a	b	0

Chromatic quasisymmetric function:

$$X_{\overrightarrow{G}}(\mathbf{x}, t) = \sum_{\text{proper } \kappa} t^{\mathrm{asc}(\kappa)} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}.$$

Example. $X_{\overrightarrow{G}}(\mathbf{x}, t) = (2 + 2t + 2t^2) M_{111} + t^2 M_{21} + M_{12}.$

Can $X_{\vec{G}}(\mathbf{x}, t)$ distinguish graphs?

By setting $t = 1$, we see that $X_{\vec{G}}(\mathbf{x}, t)$ contains more information than $X_G(\mathbf{x})$.

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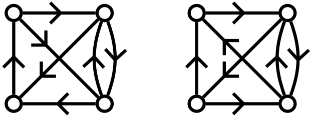

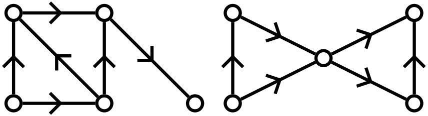

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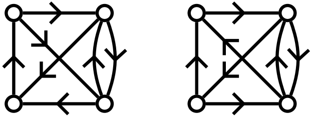

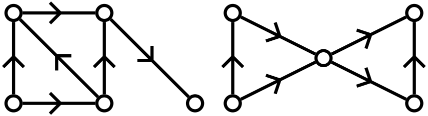

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$X_{\vec{G}}(\mathbf{x}, t)$ distinguishes directed trees. In other words, if $\vec{T}$ and $\vec{U}$ are non-isomorphic directed trees, then $X_{\vec{T}}(\mathbf{x}, t) \neq X_{\vec{U}}(\mathbf{x}, t)$.

Polls/Quizzes

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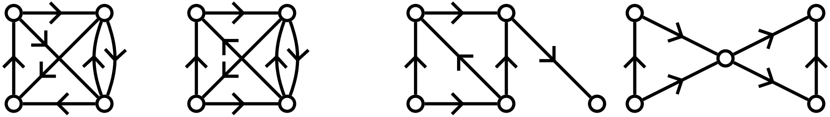

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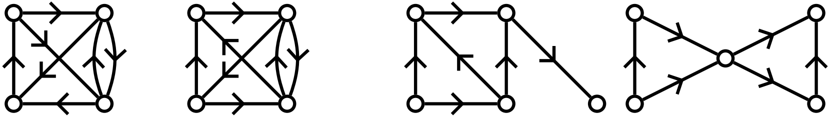

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This conjecture was our original goal.

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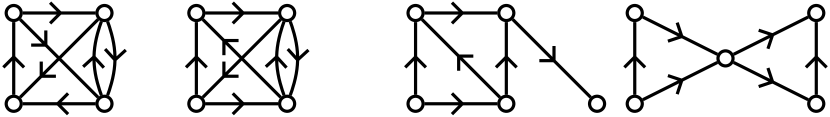

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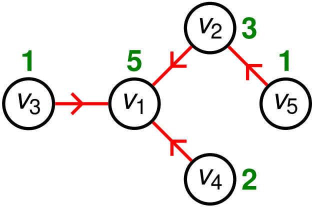
This conjecture was our original goal. **Strategy: translate to posets.**

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{\text{proper } \kappa} t^{\text{asc}(\kappa)} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}.$$

Want to show: $X_{\vec{T}}(\mathbf{x}, t) \neq X_{\vec{U}}(\mathbf{x}, t)$.

Key insight:

- Look at the coefficient of the highest power of t.
- It's enough to show these coefficients are different for T and U.
- So just look at colorings where all edges are ascents



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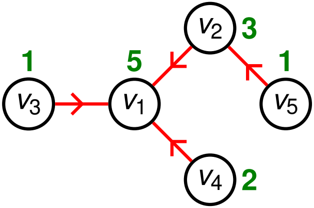
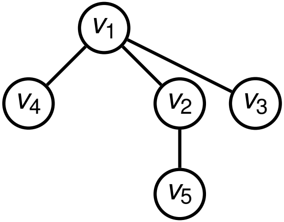
  $v_i \leq_P v_j$ if there is a directed path from v_i to v_j.





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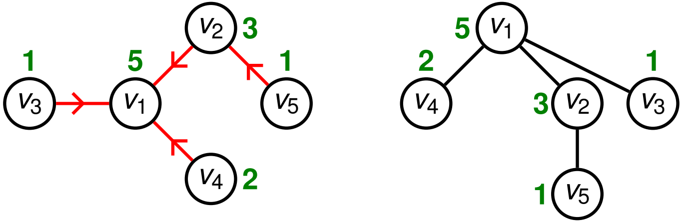

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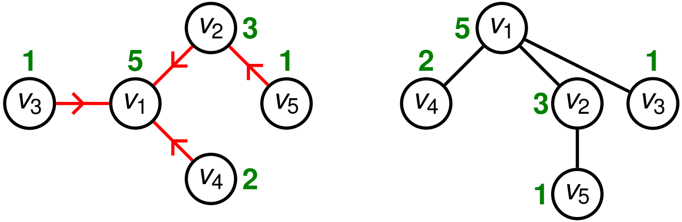
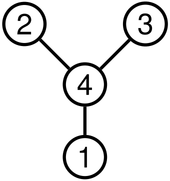
- The corresponding coloring is a strict P-partition.



Labeled posets

Labeled poset (P, ω): poset P with n elements and a bijection $\omega : P \to \{1, 2, \ldots, n\}$.

$(P, \omega) =$ 

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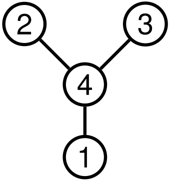
$(P, \omega) =$ 

Key definition (Stanley, 1971). A (P, ω)-partition is a map f from P to the positive integers satisfying:

- f is ordering preserving, i.e. if $a <_P b$ then $f(a) \leq f(b)$;
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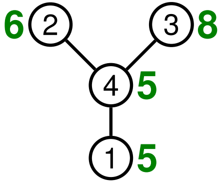
$(P, \omega) =$ 

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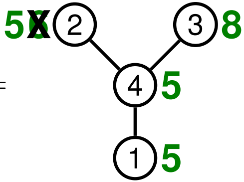
$(P, \omega) =$ 

Not a (P, ω)-partition

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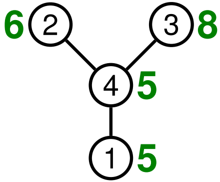
$(P, \omega) =$ 

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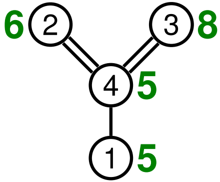
$(P, \omega) =$ 

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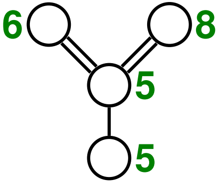
$(P, \omega) =$ 

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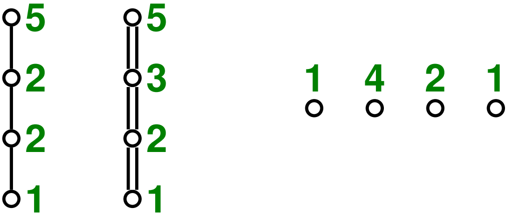
We use double edges to denote the strictness conditions and then we can (usually) ignore the underlying labeling.

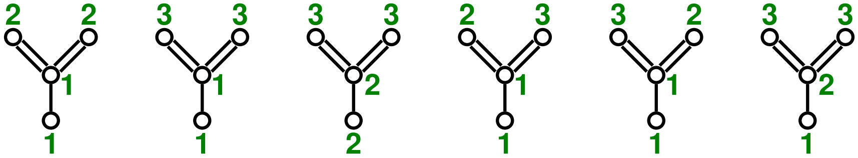
Motivating examples for (P, ω)-partitions



- (P, ω) chain with all weak edges: get a partition
- (P, ω) chain with all strict edges: get a partition with distinct parts
- (P, ω) is an antichain: get a composition

General (P, ω)-partitions interpolate between these classical objects.

Example. Resrict to $f(p) \in \{1, 2, 3\}$.



$x_1^2 x_2^2$ $\quad$ $x_1^2 x_3^2$ $\quad$ $x_2^2 x_3^2$ $\quad$ $x_1^2 x_2 x_3$ $\quad$ $x_1^2 x_2 x_3$ $\quad$ $x_1 x_2 x_3^2$

$$K_{(P,\omega)}(x_1, x_2, x_3) = x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + 2x_1^2 x_2 x_3 + x_1 x_2 x_3^2.$$

In general, the (P, ω)-partition enumerator is by given by:

$$K_{(P,\omega)}(\mathbf{x}) = \sum_{(P,\omega)\text{-partition } f} x_1^{\#f^{-1}(1)} x_2^{\#f^{-1}(2)} \cdots .$$

The (P, ω)-partition enumerator

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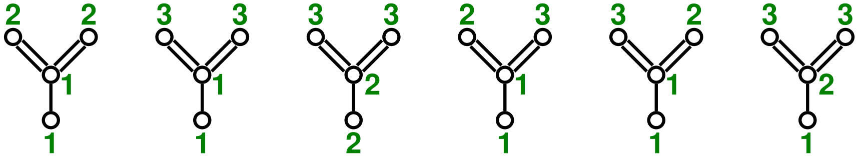

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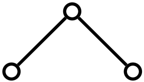
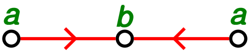
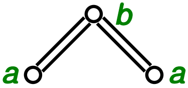
Seem familiar?



colorings of $\overrightarrow{G}$ with all ascents $\longleftrightarrow$ strict P-partitions

$a < b < c$





From $X_{\overrightarrow{G}}(\mathbf{x}, t)$ to $K_{(P,\omega)}$

colorings of $\overrightarrow{G}$ with all ascents $\longleftrightarrow$ strict P-partitions

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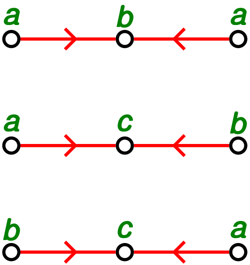

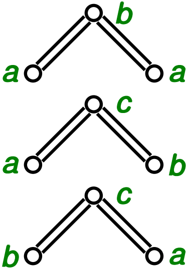

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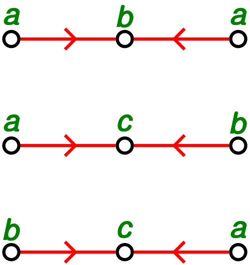

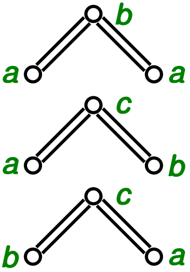

$$\text{coefficient of } t^2 \text{ in } X_{\vec{G}}(\mathbf{x}, t) \;=\; M_{21} + 2M_{111} \;=\; K_P^{<}(\mathbf{x}).$$

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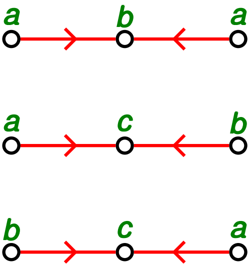

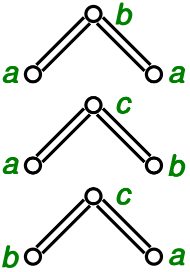

coefficient of t^2 in $X_{\vec{G}}(\mathbf{x}, t) \;=\; M_{21} + 2M_{111} \;=\; K_P^{<}(\mathbf{x})$.

For general trees, coefficient of $t^{\#E}$ in $X_{\vec{G}}(\mathbf{x}, t) \;=\; K_P^{<}(\mathbf{x})$.

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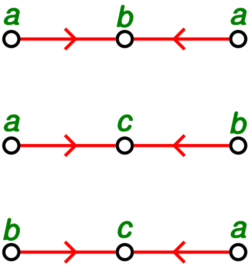

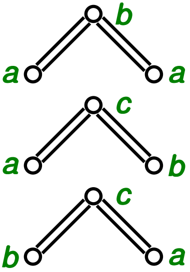

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For general trees, $\quad$ coefficient of $t^{\#E}$ in $X_{\vec{G}}(\mathbf{x}, t) \;=\; K_P^{<}(\mathbf{x})$.

Translation complete. Now study equality among $K_{(P,\omega)}(\mathbf{x})$.

From $X_{\vec{G}}(\mathbf{x}, t)$ to $K_{(P,\omega)}$

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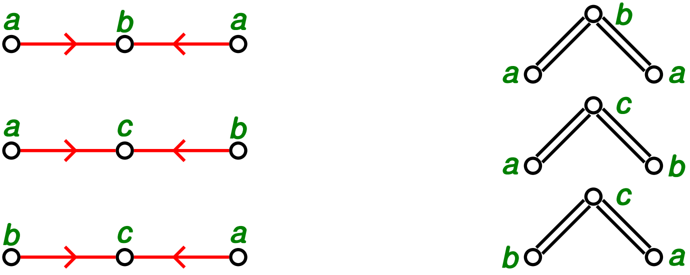

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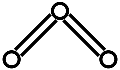
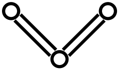
Translation complete. Now study equality among $K_{(P,\omega)}(\mathbf{x})$.
[Browning, Féray, Hasebe, Hopkins, Kelly, Liu, M., Tsujie, Ward, Weselcouch]

Can $K_{(P,\omega)}(\mathbf{x})$ distinguish posets?

Statement 5.

$K_P^<(\mathbf{x})$ distinguishes posets that are trees.
i.e. if tree posets P and Q are not isomorphic, then $K_P^<(\mathbf{x}) \neq K_Q^<(\mathbf{x})$.

Polls/Quizzes

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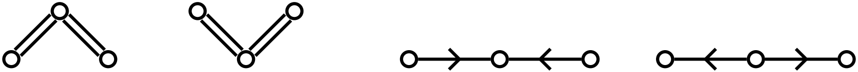

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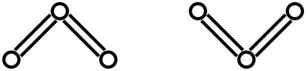

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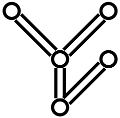

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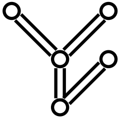

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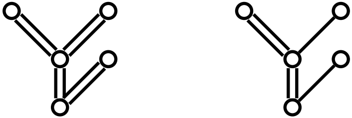

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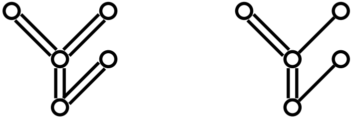

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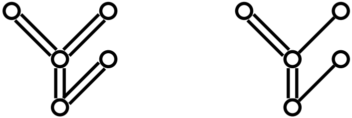

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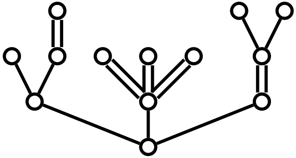
Our main contribution sits between Theorem 1 and Conjecture 4.

Fair trees and a generalization

Definition. A labeled poset that is a tree is said to be a fair tree if for each vertex, its outgoing edges up to its children are either all strict or all weak.

Example.



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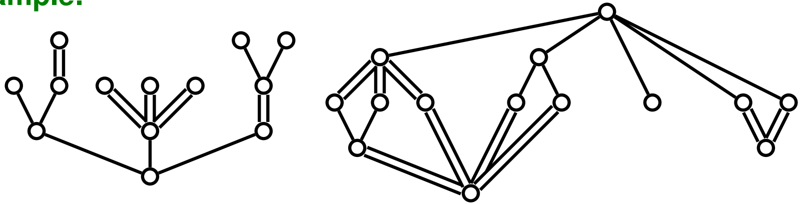

Definition. More generally, we define the set $\mathcal{C}$ of labeled posets recursively by:

1. the one-element labeled poset [1] is in $\mathcal{C}$;
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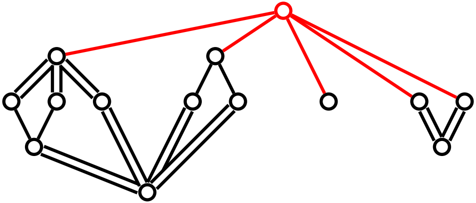

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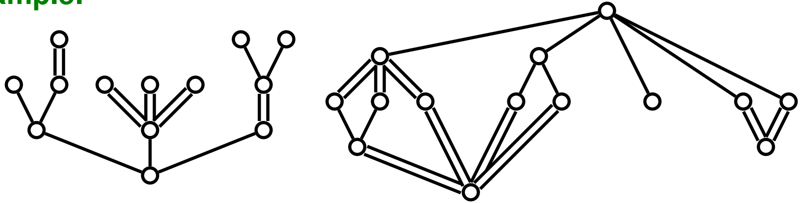

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Theorem 2 [Aval, Djenabou, M., 2022].
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Irreducibility is also the crux for

- Hasebe & Tsujie;
- Ricki Ini Liu & Michael Weselcouch ($K_P^{<}(\mathbf{x})$ distinguishes series-parallel posets; needs irreducibility for general connected P with all strict edges, 2020).

Main tool in this research area

Stanley, 1971 and Ira Gessel, 1984:
$K_{(P,\omega)}(\mathbf{x})$ expands beautifully in F-basis.

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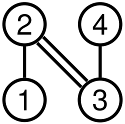

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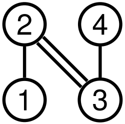

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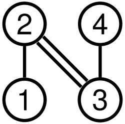

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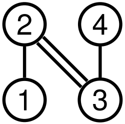

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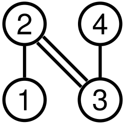

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Theorem [Gessel & Stanley]. For a labeled poset (P,ω),

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Surprising Conjecture 5 [Nick Loehr & Greg Warrington, 2022]. $X_G(1, q, q^2, \ldots, q^{n-1})$ distinguishes trees with n vertices, i.e. if T and U are non-isomorphic trees with n vertices, then

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Thanks for your attention!