Tiling puzzles

Peter McNamara

Student Colloquium Series
Bucknell University
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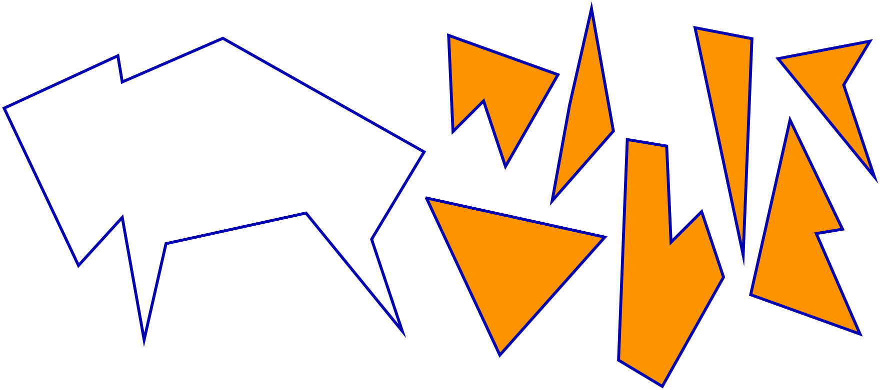
Slides available from
www.facstaff.bucknell.edu/pm040/research.html

What is a tiling?

Tangrams:



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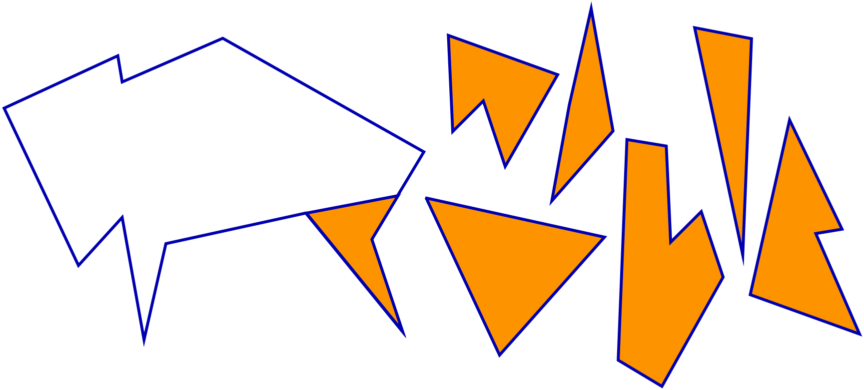

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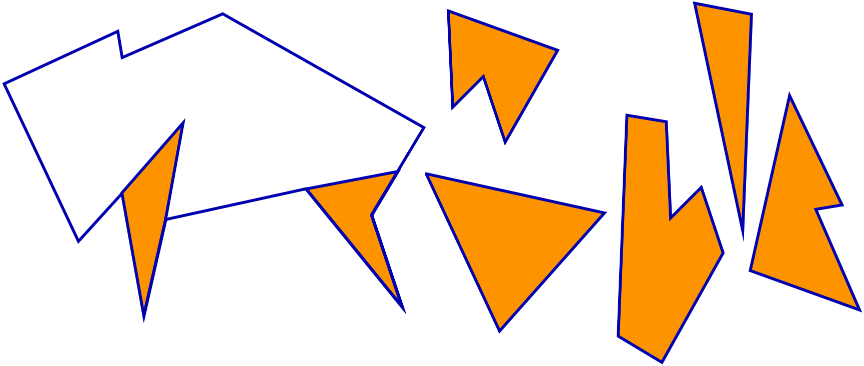

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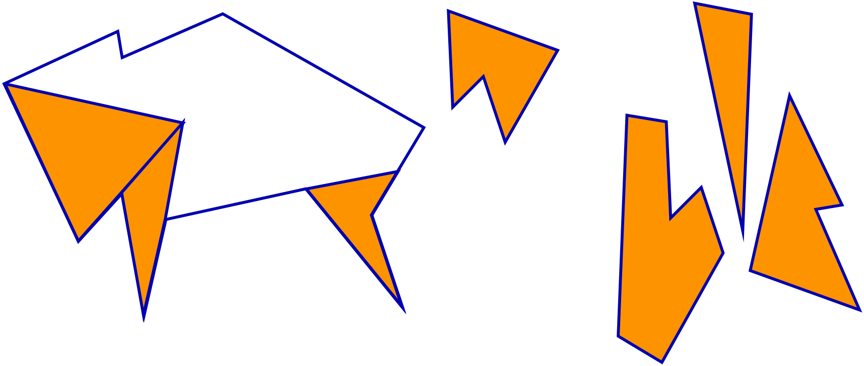

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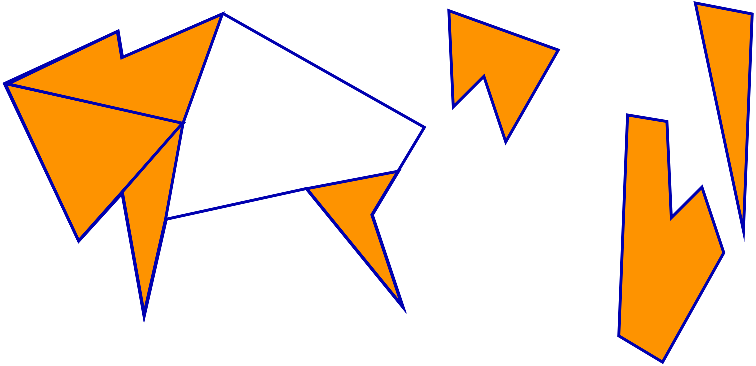

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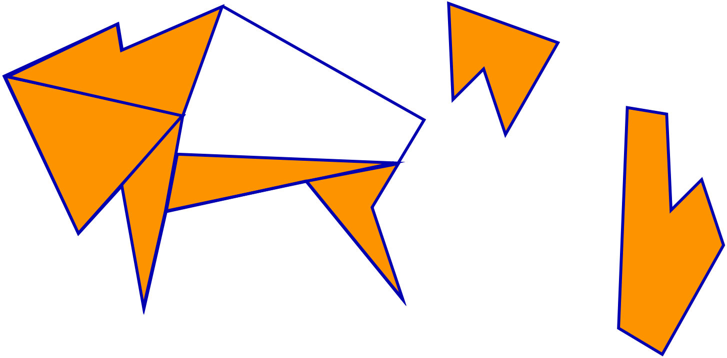

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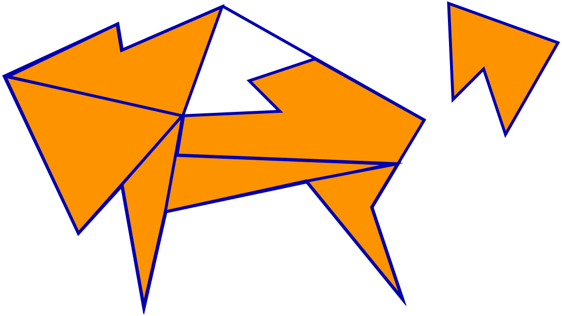

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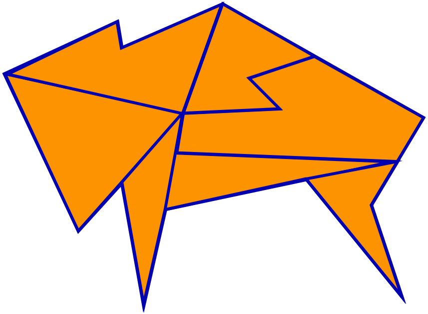

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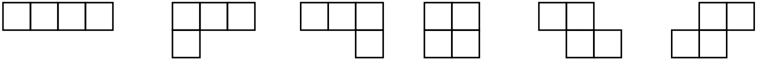
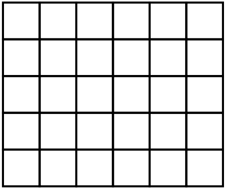
Based on an expository paper of Richard Stanley and Federico Ardila.

Is there a tiling?

Tetris pieces:



Can we tile a 6×5 rectangle with the tetris pieces, using each piece as many times as we like?



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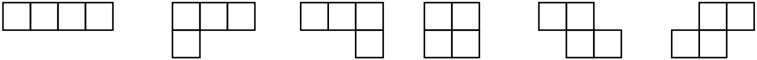

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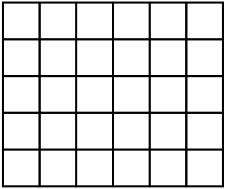

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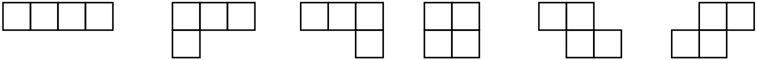

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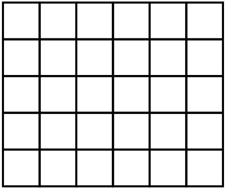
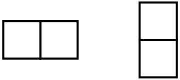


No.

Each piece has 4 boxes.
There are 30 boxes to fill.
4 does not divide into 30 evenly. (Divisibility argument)

Is there a tiling of a chessboard with dominoes?

Dominoes:



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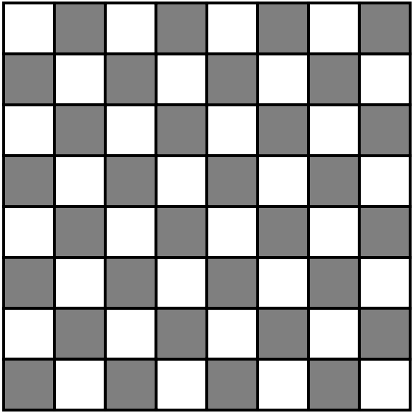

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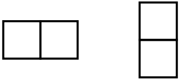

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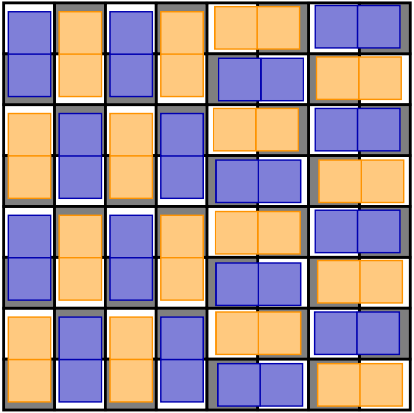
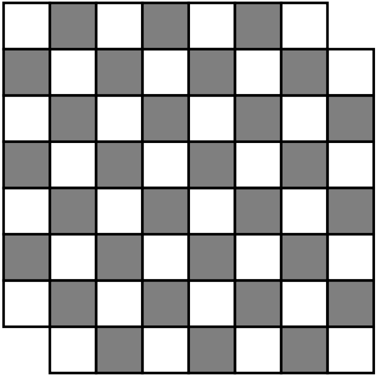


Is there a tiling of a holey chessboard?

Can we tile a this modified chessboard with dominoes?
62 squares: 30 black, 32 white.



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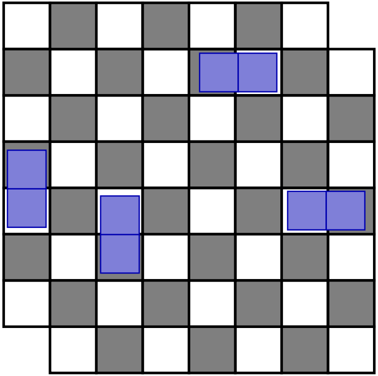

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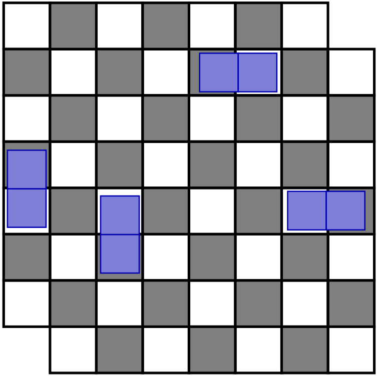
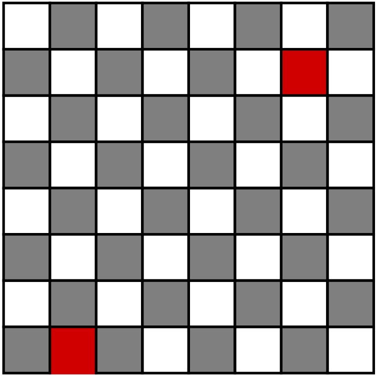


Every domino covers exactly one black square and one white square. But there are not the same number of white squares as black squares. (Coloring argument)

Is there a tiling of a fair holey chessboard?

What if we remove 1 black and 1 white square?
62 squares: 31 black, 31 white.



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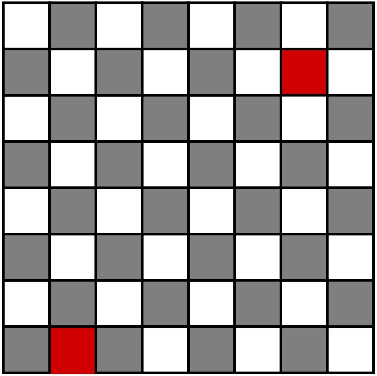

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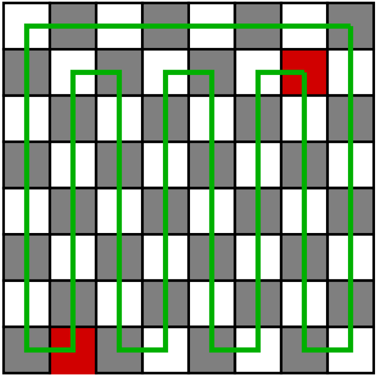

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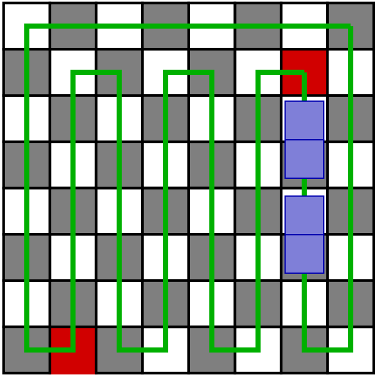

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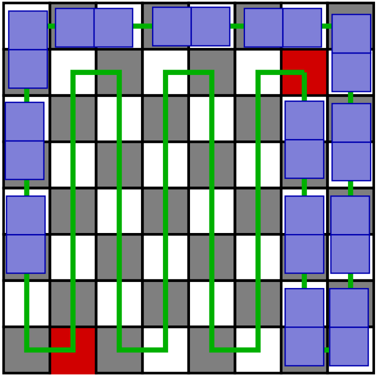

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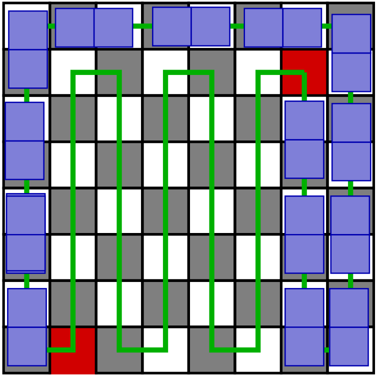

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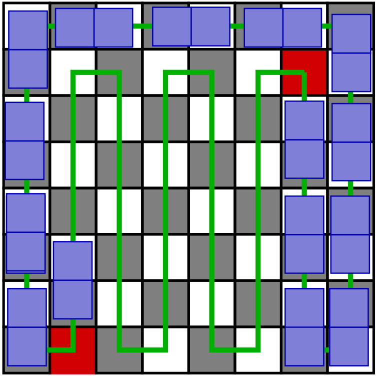

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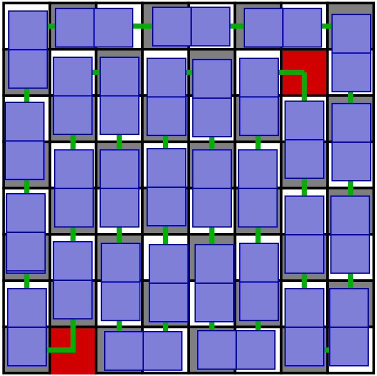

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What if we remove any 2 black and any 2 white squares?
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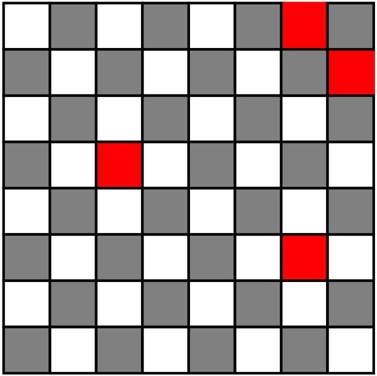

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Fisher & Temperley, Kasteleyn (independently, 1961):
The number of tilings of a $2m \times 2n$ rectangle with dominoes is

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

For example, for a chessboard $m = n = 4$, and we get

$$4^{16} \prod_{j=1}^{4} \prod_{k=1}^{4} \left(\cos^2 \frac{j\pi}{9} + \cos^2 \frac{k\pi}{9} \right).$$

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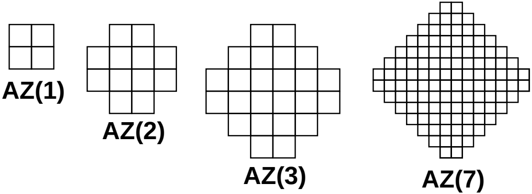
Answer = 12,988,816 .

How many tilings of Aztec diamonds with dominoes?



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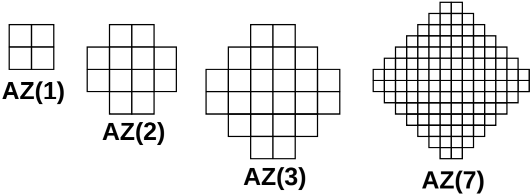
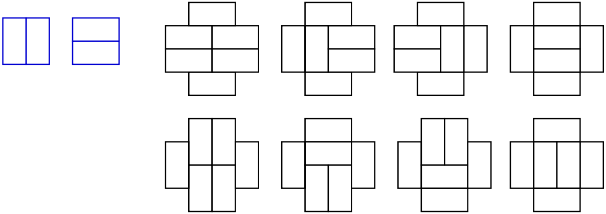


Tilings with dominoes:



How many tilings of Aztec diamonds (continued)

$2, 8, 64, 1024, \ldots$.

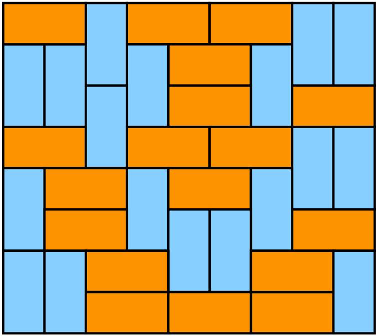
Elkies, Kuperberg, Larsen & Propp (1992):
In general, AZ(n) has $2^{\frac{n(n+1)}{2}}$ tilings with dominoes. (4 proofs)

Now around 12 proofs, but none are really simple.

Open Problem

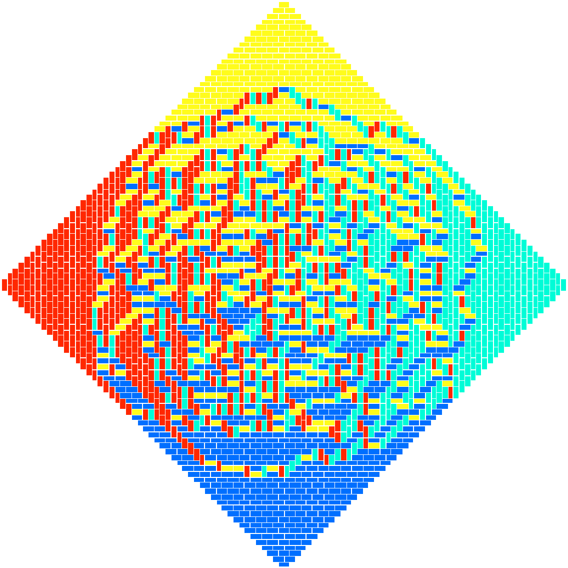
Find a simple proof that the number of tilings of AZ(n) is $2^{\frac{n(n+1)}{2}}$.

What does a typical tiling look like?



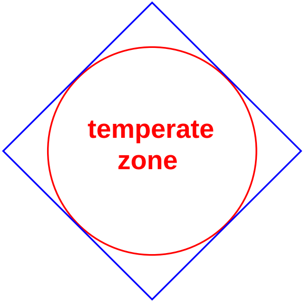
No obvious structure.
But if we work with Aztec diamonds....

A typical tiling of AZ(50)



Jockusch, Propp and Shor, 1995.

The Arctic Circle Theorem. Fix $\varepsilon > 0$. Then for all sufficiently large *n*, all but an ε fraction of the domino tilings of AZ(*n*) will have a temperate zone whose boundary stays uniformly within distance εn of the inscribed circle.



In other words: usually, almost everything outside the circle is “frozen” in place.

Similar phenomena observed for other cases.

“To infinity and beyond” – Lightyear, Buzz, 1995

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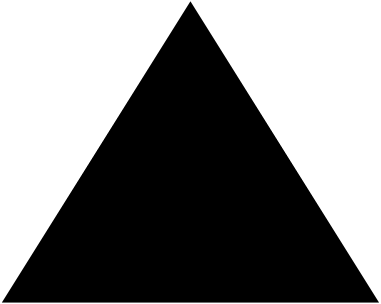
Sierpinski triangle:



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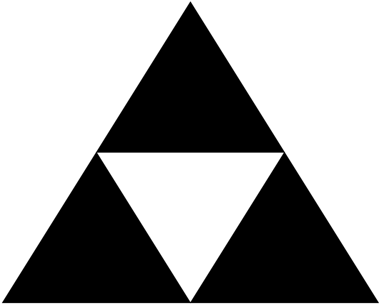

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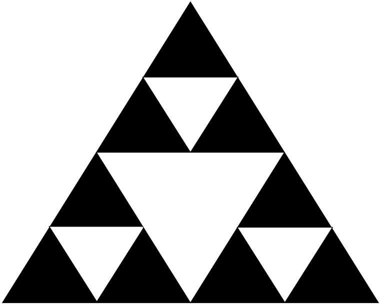

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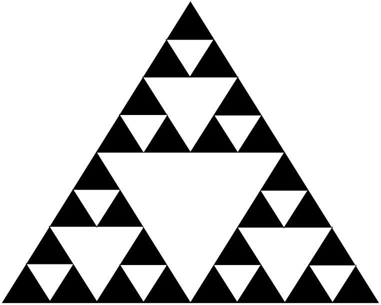

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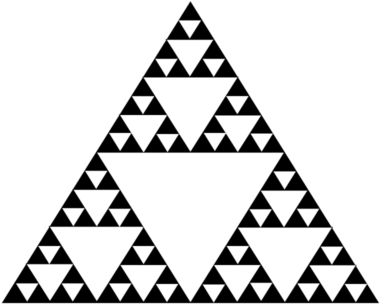

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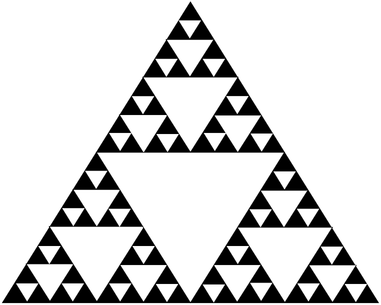

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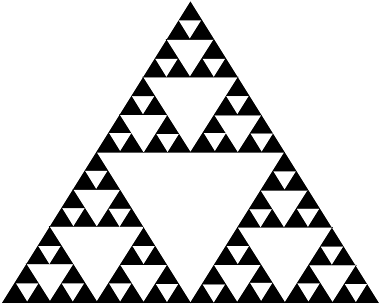

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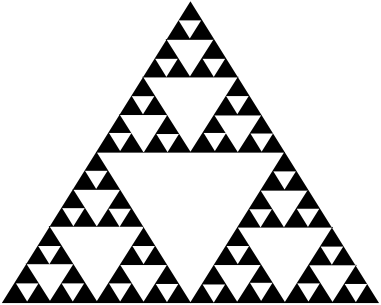

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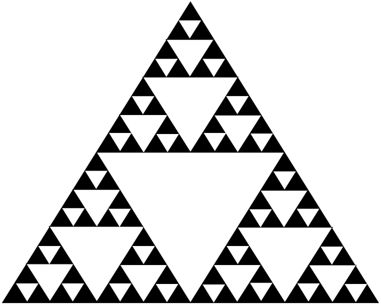

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The Sierpinski triangle is very fashionable:

Sierpinski triangle side comment

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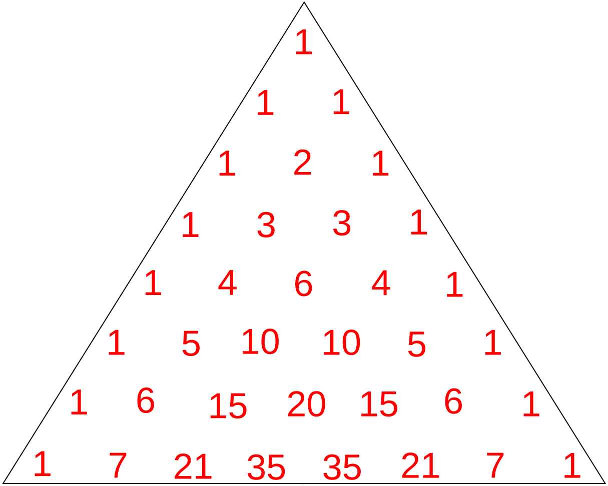
Designer: Eri Matsui

Another Sierpinski triangle side comment

Another famous triangle is Pascal's triangle.
Take the first 2^n rows:



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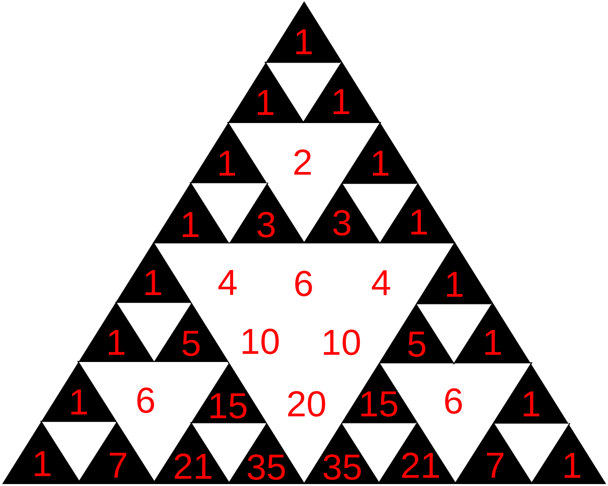

From a series to a tiling

Calc 2: $$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \cdots \quad = \quad 1.$$

From a series to a tiling

Calc 2:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots = 1.$$

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots = 1.$$

From a series to a tiling

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$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots = 1.$$
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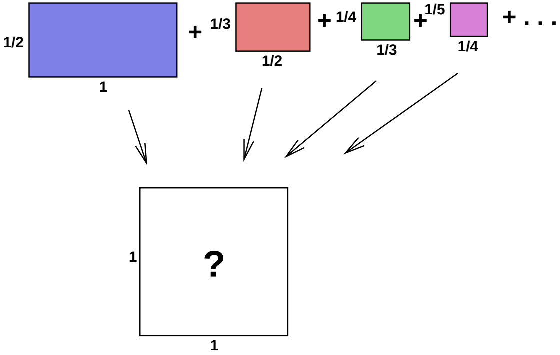

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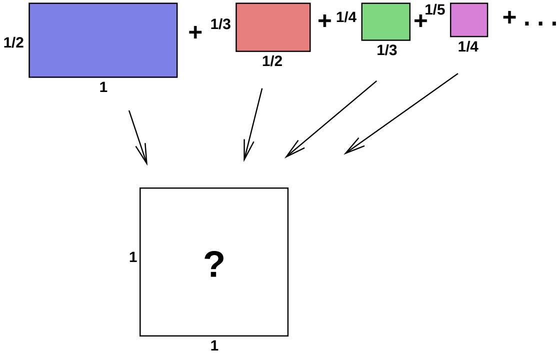

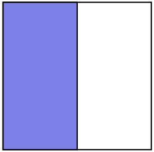

From a series to a tiling

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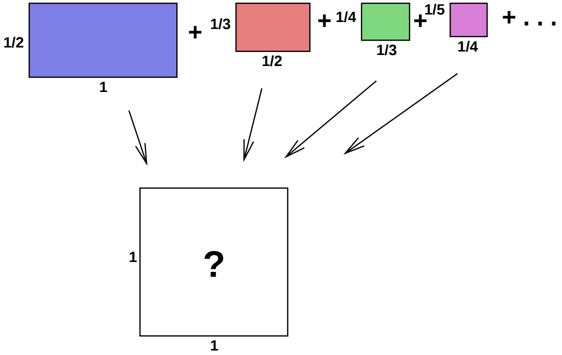

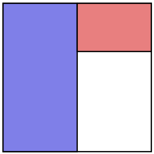

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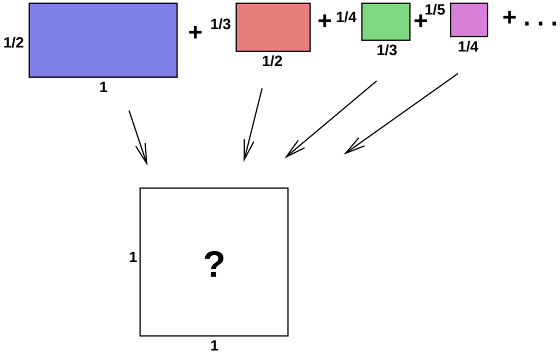

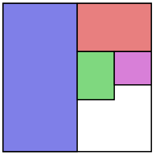

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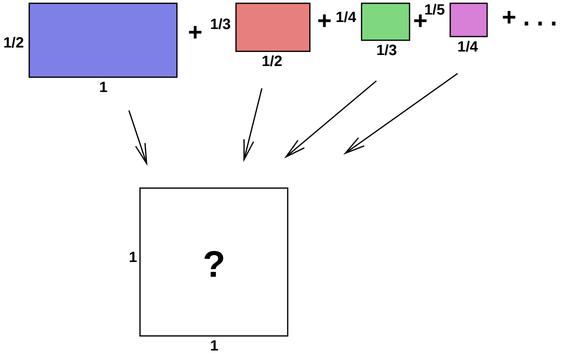
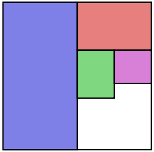




Open Problem

Find a way to tile the whole region, or show that no tiling exists.

Tiling infinite regions

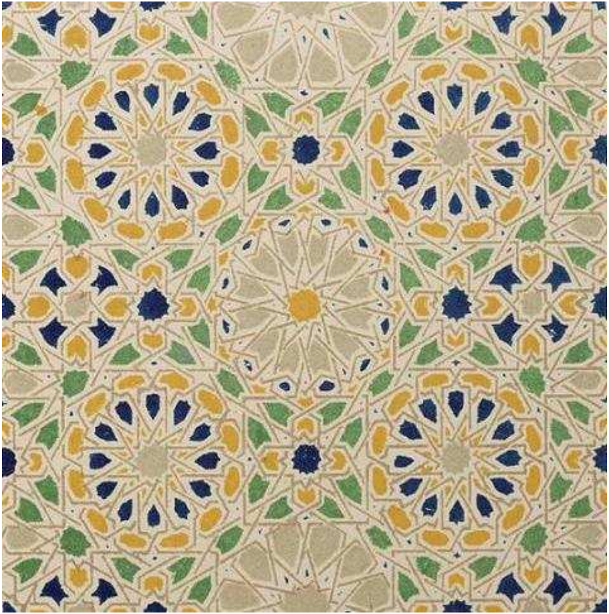
Alhambra palace, Granada, Spain.

Tiling infinite regions

Alhambra palace, Granada, Spain.

Abstract Algebra: There are essentially 17 different tiling patterns of the plane that have translation symmetries in two different directions. *Plane crystallographic groups / wallpaper groups*

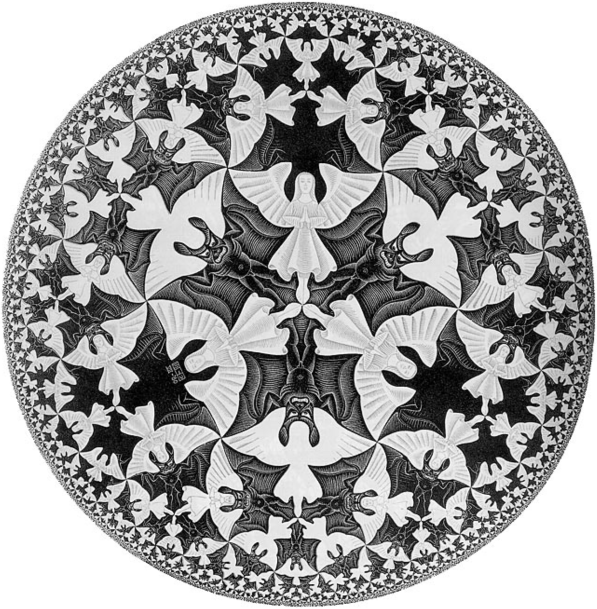
Another Alhambra tiling



Escher tilings

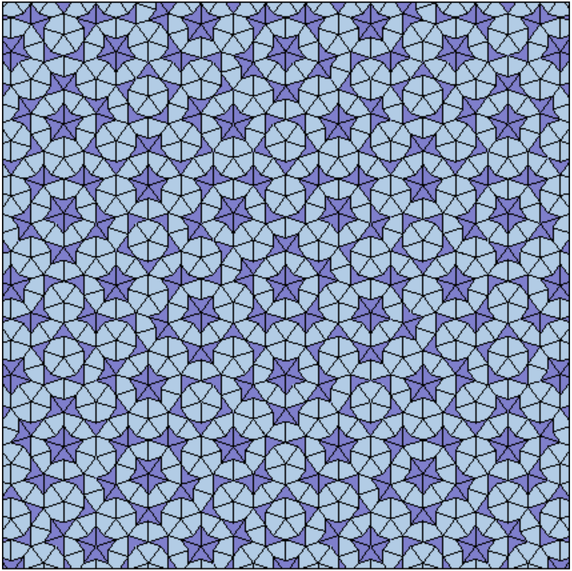
Maurits Cornelis Escher (1898-1972): *Although I am absolutely without training in the exact sciences, I often seem to have more in common with mathematicians that with my fellow artists.*

Another Escher tiling

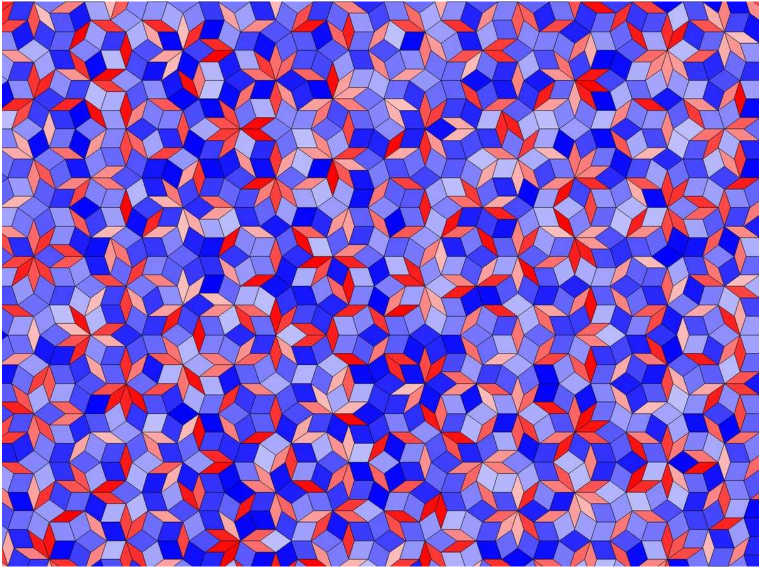


Opposite direction: no symmetry at all!

Sir Roger Penrose



Another Penrose tiling



Conference on Undergraduate Research in Mathematics

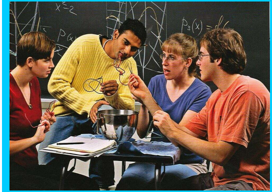
Penn State University, November 9-10, 2007

Plenary speakers:

- **George Andrews, Penn State University**
- **Frank Morgan, Williams College**

Funding for the conference provided by:

- **Penn State's Eberly College of Science**
- **Penn State's Department of Mathematics**
- **Penn State's Women in Mathematics Program**
- **The Mathematical Association of America through NSF grant DMS-0241090**



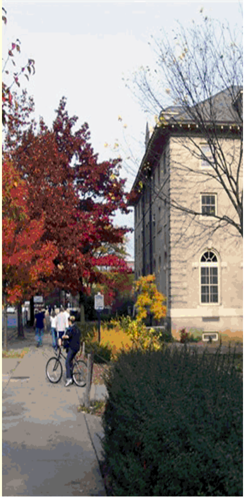
This conference will showcase mathematical research by undergraduate students through 15-minute contributed talks and posters.

The conference social program includes:

- A Friday evening reception and dinner
- A Friday evening stroll to the famous Penn State Berkey Creamery for ice cream
- Breakfast and lunch on Saturday
- Tours of the Mathematics Department's facilities, including the William Pritchard Fluid Mechanics Lab and the SCREMS Lab
- Lots of coffee and cookie breaks!

To register, go to:
www.math.psu.edu/ug/curm/conference07/

For more information about the conference, contact the co-coordinators, James Sellers and Diane Henderson, at: curm@math.psu.edu



Contact `peter.mcnamara@bucknell.edu` or `emily.dryden@bucknell.edu`.