The Combinatorial Topology of the Permutation Pattern Poset

Peter McNamara
Bucknell University

Joint work with:
Einar Steingrímsson
University of Strathclyde

Bijective and Algebraic Combinatorics
in honor of Bruce Sagan's 60th birthday

25 March 2014

Slides and paper available from
`www.facstaff.bucknell.edu/pm040/`

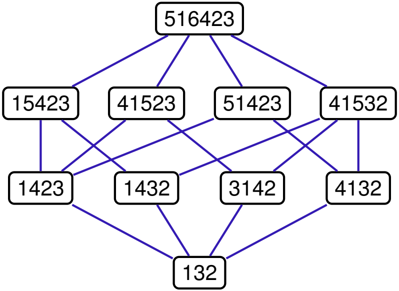
Outline

- The PPP setting
- Some combinatorial topology
- Outline of results
- Open problems

Motivation: Wilf's question

Pattern order: order permutations by pattern containment.

e.g., $4132 \leq 516423$



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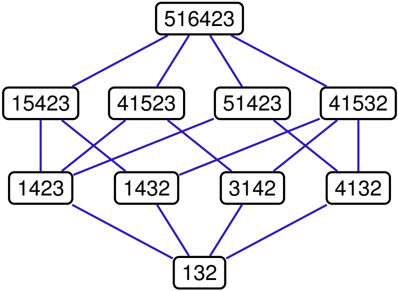

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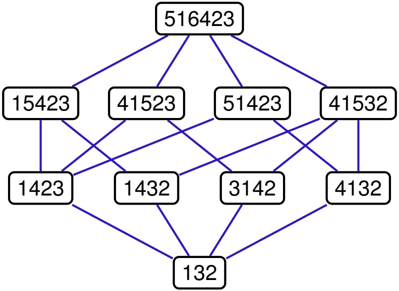

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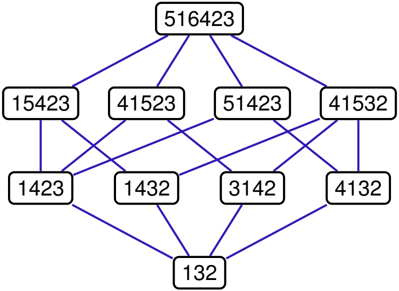

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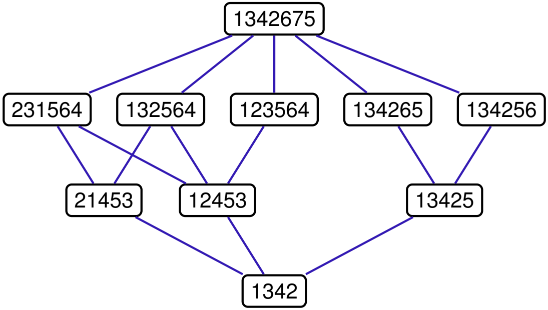
Still open.

Other topological questions

- Are the open intervals connected?
- Shellable?
- What is their homotopy type?



Some combinatorial topology

Poset $P \longrightarrow$ Simplicial complex $\Delta(P)$

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Order complex of $[p, q]$: faces of $\Delta(p, q)$ are the chains in (p, q).

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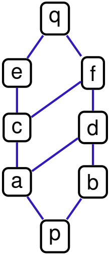

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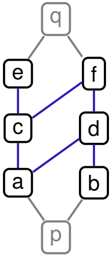

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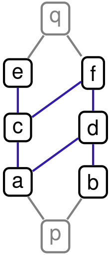

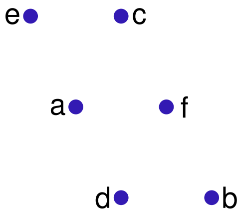

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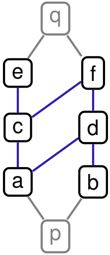

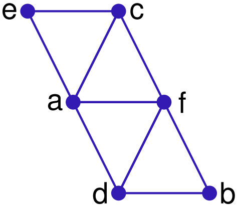

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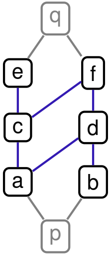

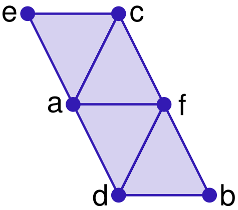

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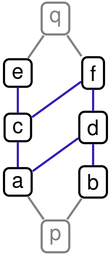

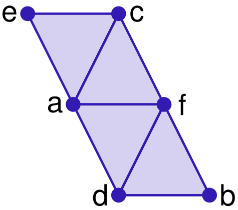

Definition. A pure d-dimensional complex is shellable if its facets can be ordered $F_1, F_2, \ldots, F_n$ such that, for all $2 \le i \le n$,

$$F_i \cap (F_1 \cup F_2 \cup \cdots \cup F_{i-1})$$

is pure and $(d-1)$-dimensional.

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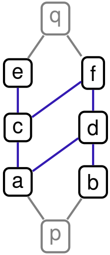

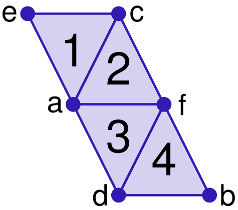

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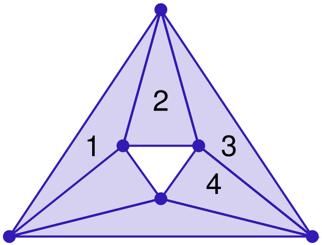
- 

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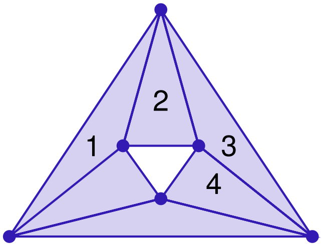
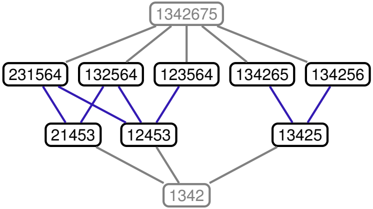
- 

- Main counterexample. (p, q) disconnected with $d \geq 1$: $\Delta(p, q)$ is not shellable.



The interval above is said to be non-trivially disconnected.

Why we care about shellability

- If $\Delta(p, q)$ is shellable, then its either contractible, or homotopic to a wedge of $|\mu(p, q)|$ spheres in the top dimension.
- Combinatorial tools for showing shellability of $\Delta(P)$: EL-shellability, CL-shellability, etc.

Results

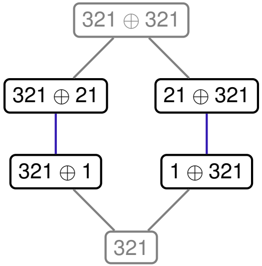
Direct sum: $21 \oplus 3214 = 215436$.

Skew sum: $21 \ominus 3214 = 653214$.

π is indecomposable if $\pi \neq \alpha \oplus \beta$ for any non-empty α, β.

Lemma. If π is indecomposable with $|\pi| \geq 3$, then $\Delta(\pi, \pi \oplus \pi)$ is disconnected and so not shellable.

Example.



Almost all intervals are not shellable

Theorem [McN. & Steingrímsson].
Fix σ. Randomly choose τ of length n.

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Idea.

- Björner: If $[\sigma,\tau]$ is shellable (i.e. $\Delta(\sigma,\tau)$ is), then so is every subinterval of $[\sigma,\tau]$.
- Thus, if $[\sigma,\tau]$ contains a (non-trivial) disconnected subinterval, then it can't be shellable.
- Show every $[\sigma,\tau]$ as $n\to\infty$ contains $[\pi,\pi\oplus\pi]$ with π indecomposable,
  or contains $[\pi,\pi\ominus\pi]$ with π skew indecomposable.

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Several other results about disconnected intervals.

Shellable intervals

In contrast, there's a large class of shellable intervals.

Definition. π is layered if it takes the form $\pi = \pi^1 \oplus \pi^2 \oplus \cdots \oplus \pi^k$ with each π^i decreasing.

e.g. $1 \oplus 1 \oplus 321 \oplus 321 \oplus 1 \; = \; 125438769 \; = \; 11331$.

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Lemma. In layered case, it's trivial to check if $[\sigma, \tau]$ is disconnected.

Theorem [McN. & Steingrímsson]. Suppose σ, τ layered such that $[\sigma, \tau]$ does not contain a non-trivial disconnected subinterval. Then $[\sigma, \tau]$ is shellable.

Example. $[1 \oplus 321 \oplus 1, \;\; 321 \oplus 321 \oplus 1 \oplus 21 \oplus 1] = [131, 33121]$ is shellable.

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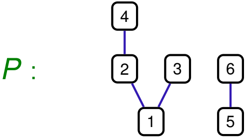
Idea of proof. Show $[\sigma, \tau]$ is dual CL-shellable.

Two connections to generalized subword order.

P: any poset.

P^*: set of words over the alphabet P.



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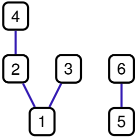

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Main Definition. $u \le w$ if there exists a subword $w(i_1)w(i_2)\cdots w(i_r)$ of w of the same length as u such that

$$u(j) \le_P w(i_j) \text{ for } 1 \le j \le r.$$

Example. P is the chain $1 < 2 < 3 < 4 < \cdots$
gives containment order for layered permutations.

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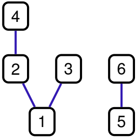

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Connection 1.

Theorem [McN. & Steingrímsson]. If *P* is a rooted forest, then $[u, w]$ is shellable iff it does not contain a non-trivial disconnected subinterval.

A Möbius function formula

$\sigma = \sigma_1 \oplus \cdots \oplus \sigma_s$, the finest decomposition of σ.
$\tau = \tau_1 \oplus \cdots \oplus \tau_t$, the finest decomposition of τ.

Burstein, Jelínek, Jelínková & Steingrímsson:
2 propositions for expressing $\mu(\sigma, \tau)$ in terms of $\mu(\sigma_i, \tau_j)$.

Theorem [McN. & Steingrímsson].

$$\mu(\sigma,\tau) = \sum_{\sigma=\sigma_1\oplus\cdots\oplus\sigma_t} \prod_{1\leq m\leq t} \begin{cases} \mu(\sigma_m, \tau_m) + 1 & \text{if } \sigma_m = \emptyset \text{ and } \tau_{m-1} = \tau_m, \\ \mu(\sigma_m, \tau_m) & \text{otherwise,} \end{cases}$$

where the sum is over all direct sums $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_t$ such that $\emptyset \leq \sigma_m \leq \tau_m$ for all $1 \leq m \leq t$.

Connection 2. This is identical to the formula for μ for generalized subword order: replace indecomposable parts by letters from P.

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Open problem. Why?

More open problems

- Understand non-shellable intervals without disconnected subintervals.
  e.g. $[123, 3416725]$.
- Find a good way to test shellability by computer.
- Separable permutations: can be built from 1 by a sequence of direct sums or skew sums.

$$\begin{aligned} 1 \oplus 1 &= 12 \\ (1 \oplus 1) \ominus (1 \oplus 1) &= 12 \ominus 12 = 3412 \\ 1 \oplus 3412 &= 14523 \text{ etc.} \end{aligned}$$

  Conjecture. Suppose σ, τ separable such that $[\sigma, \tau]$ does not contain a non-trivial disconnected subinterval. Then $[\sigma, \tau]$ is shellable.
- Conjecture. $[\sigma, \tau]$ is always rank unimodal.

Consecutive pattern poset

Joint with Sergi Elizalde.

Consecutive pattern poset: $\sigma \leq \tau$ if σ appears as a set of consecutive letters in τ.
e.g. $213 \leq 254613$.

Möbius function: Bernini–Ferrari–Steingrímsson, Sagan–Willenbring

Theorem [Sagan & Willenbring]. Any interval is homoptic to a sphere or is contractible.

Theorems [Elizalde & McN.]

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