Comparing skew Schur functions: a quasisymmetric perspective

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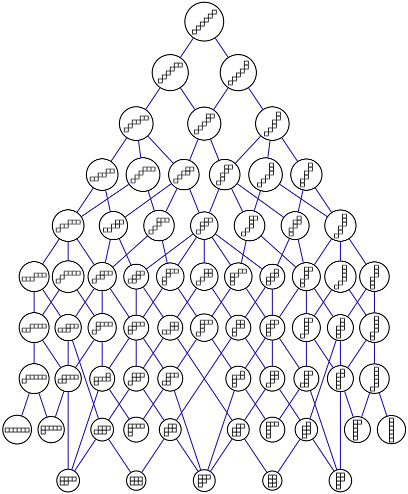
Slides and paper available from
`www.facstaff.bucknell.edu/pm040/`

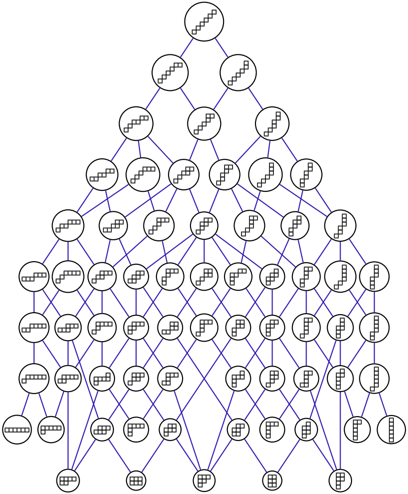
Outline

- The background story: the equality question
- Conditions for Schur-positivity
- Quasisymmetric insights and the main conjecture

Preview



F-support containment



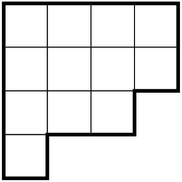
Dual of row overlap dominance

Schur functions

Cauchy, 1815

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$

- Young diagram.
  Example:
  $\lambda \quad = (4, 4, 3, 1)$



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$\leqslant$

1	3	3	4
4	4	4	9
5	6	6	
7			

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$$\begin{array}{c} \leqslant \\ \wedge \; \begin{array}{|c|c|c|c|} \hline 1 & 3 & 3 & 4 \\ \hline 4 & 4 & 4 & 9 \\ \hline 5 & 6 & 6 \\ \cline{1-3} 7 \\ \cline{1-1} \end{array} \end{array}$$

The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

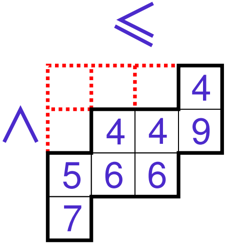
$$s_\lambda \quad = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

Example.
$s_{4431} \quad = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \cdots .$

Skew Schur functions

Cauchy, 1815

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- μ fits inside λ.
- Young diagram.
  Example:
  $\lambda/\mu = (4,4,3,1)/(3,1)$
- Semistandard Young tableau (SSYT)



The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

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$s_{4431/31} = \quad x_4^3 x_5 x_6^2 x_7 x_9 + \cdots .$

The beginning of the story

s_A: the skew Schur function for the skew shape A.

Key Facts.

- s_A is symmetric in the variables $x_1, x_2, \ldots$.
- The (non-skew) s_λ form a basis for the symmetric functions.

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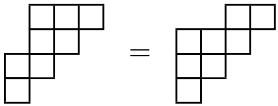
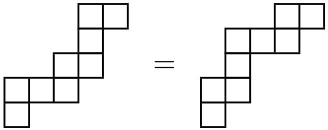
Wide Open Question. When is $s_A = s_B$?
Determine necessary and sufficient conditions on shapes of A and B.





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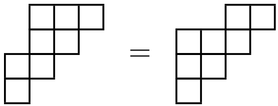

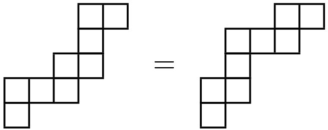

- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004)
- John Stembridge (2004)
- Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006)
- McN., Steph van Willigenburg (2006)
- Christian Gutschwager (2008)

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Definition [Reiner, Shaw, van Willigenburg]. For a skew shape A, let $\text{overlap}_k(i)$ be the number of columns occupied in common by rows $i, i+1, \ldots, i+k-1$.
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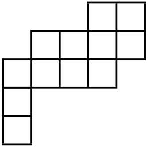
$A =$ 

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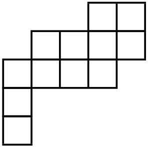
$A =$ 

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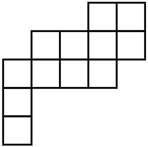
$A =$ 

- $\text{overlap}_1(i) =$ length of the ith row. Thus $\text{rows}_1(A) = 44211$.
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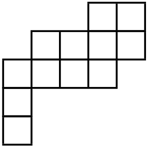
$A =$ 

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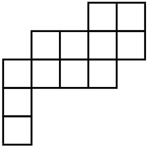
$A =$ 

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- $\mathrm{rows}_3(A) = 11$.
- $\mathrm{rows}_k(A) = \emptyset$ for $k > 3$.

Necessary conditions for equality

Theorem [RSvW, 2006]. Let A and B be skew shapes.
If $s_A = s_B$, then

$$\mathrm{rows}_k(A) = \mathrm{rows}_k(B) \ \text{ for all } k.$$

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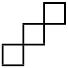
Example. $A =$ 

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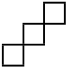
Example. $A =$ 

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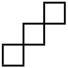
Example. $A =$ 

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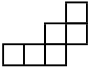
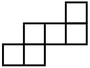
Converse is definitely not true:  $\neq$ 

Schur-positivity order

Our interest: inequalities.

Skew Schur functions are Schur-positive:

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}.$$

Original Question. When is $s_{\lambda/\mu} - s_{\sigma/\tau}$ Schur-positive?

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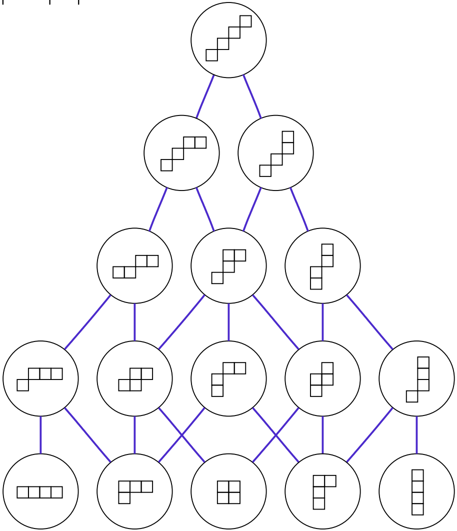
Definition. Let A, B be skew shapes. We say that

$$A \geq_s B \quad \text{if} \quad s_A - s_B \quad \text{is Schur-positive.}$$

Original goal: Characterize the Schur-positivity order $\geq_s$ in terms of skew shapes.

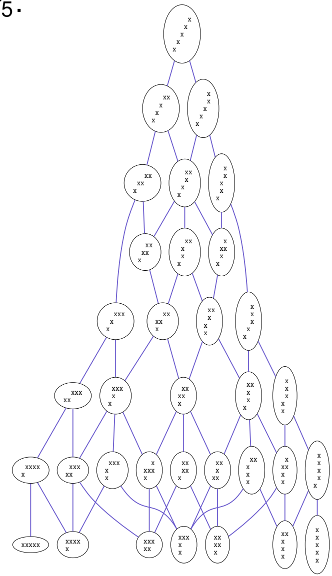
Example of a Schur-positivity poset

If $B \leq_s A$ then $|A| = |B|$.
Call the resulting ordered set P_n.
Then P_4:

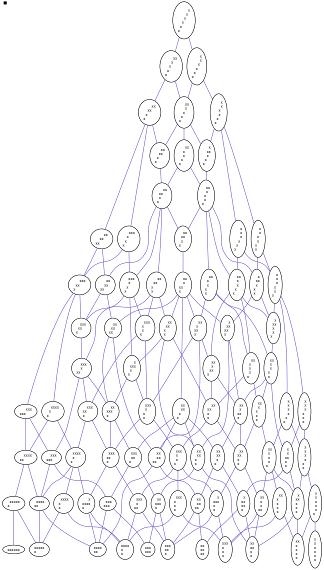


More examples

P_5:



P_6:



Known properties: Sufficient conditions

Sufficient conditions for $A \geq_s B$:

- Alain Lascoux, Bernard Leclerc, Jean-Yves Thibon (1997)
- Andrei Okounkov (1997)
- Sergey Fomin, William Fulton, Chi-Kwong Li, Yiu-Tung Poon (2003)
- Anatol N. Kirillov (2004)
- Thomas Lam, Alex Postnikov, Pavlo Pylyavskyy (2005)
- François Bergeron, Riccardo Biagioli, Mercedes Rosas (2006)
- McN., Steph van Willigenburg (2009, 2012)
- ...

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Notation. Write $\lambda \preccurlyeq \mu$ if λ is less than or equal to μ in dominance order, i.e.

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Theorem [McN. (2008)]. Let A and B be skew shapes. If $s_A - s_B$ is Schur-positive, then

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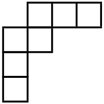
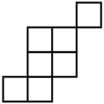
$A =$  $B =$ 

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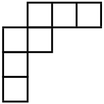
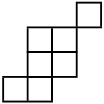
$A =$  $B =$ 

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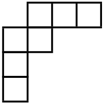
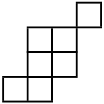
$A =$  $B =$ 

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So A and B are incomparable in Schur-positivity poset (and in "Schur support containment poset").

Summary so far

$$\boxed{s_A - s_B \text{ is Schur-pos.}} \quad \Rightarrow \quad \boxed{\operatorname{supp}_s(A) \supseteq \operatorname{supp}_s(B)} \quad \Rightarrow \quad \boxed{\begin{array}{l} \operatorname{rows}_k(A) \preccurlyeq \operatorname{rows}_k(B)\ \forall k \\ \operatorname{cols}_\ell(A) \preccurlyeq \operatorname{cols}_\ell(B)\ \forall \ell \\ \operatorname{rects}_{k,\ell}(A) \leq \operatorname{rects}_{k,\ell}(B)\ \forall k, \ell \end{array}}$$

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MIND THE GAP

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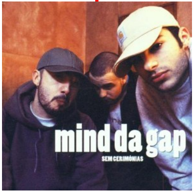

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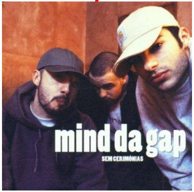
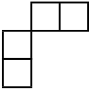
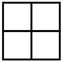
$\uparrow$



Converse is very false.

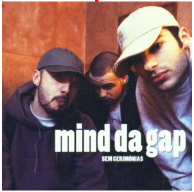
Example.

$A =$  $\qquad B =$ 

$s_A = s_{31} + s_{211} \qquad s_B = s_{22}$

Summary so far

$\boxed{s_A - s_B \text{ is Schur-pos.}}$ $\Rightarrow$ $\boxed{\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)}$ $\Rightarrow$ $\boxed{\begin{array}{l} \mathrm{rows}_k(A) \preccurlyeq \mathrm{rows}_k(B)\ \forall k \\ \mathrm{cols}_\ell(A) \preccurlyeq \mathrm{cols}_\ell(B)\ \forall \ell \\ \mathrm{rects}_{k,\ell}(A) \leq \mathrm{rects}_{k,\ell}(B)\ \forall k, \ell \end{array}}$



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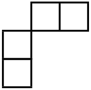
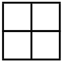
$A =$  $B =$ 

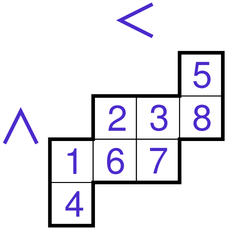
$s_A = s_{31} + s_{211}$ $s_B = s_{22}$

New Goal: Find weaker algebraic conditions on A and B that imply the overlap conditions.

What algebraic conditions are being encapsulated by the overlap conditions?

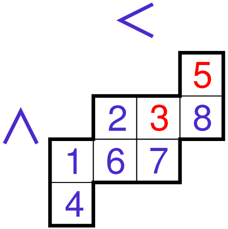
Answer: use the F-basis of quasisymmetric functions

- Skew shape A.
- Standard Young tableau (SYT) T of A.



Answer: use the F-basis of quasisymmetric functions

- Skew shape A.
- Standard Young tableau (SYT) T of A.
- Descent set: $S(T) = \{3, 5\}$.



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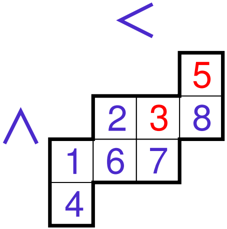

Then s_A expands in the basis of fundamental quasisymmetric functions as

$$s_A = \sum_{\text{SYT } T} F_{S(T)}.$$

Example.

$$s_{4431/31} = F_{\{3,5\}} + \cdots .$$

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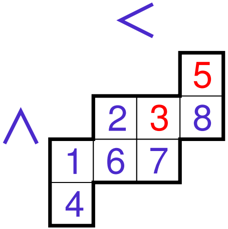

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Example.

$$s_{4431/31} = F_{\{3,5\}} + \cdots .$$

Facts.

- The F form a basis for the quasisymmetric functions.
- So notions of F-positivity and F-support make sense.
- Schur-positivity implies F-positivity.
- $\text{supp}_s(A) \supseteq \text{supp}_s(B)$ implies $\text{supp}_F(A) \supseteq \text{supp}_F(B)$

New results: filling the gap

Theorem. [McN. (2013)]

$$
\begin{array}{ccccc}
\boxed{s_A - s_B \text{ is Schur-pos.}} & \Rightarrow & \boxed{\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)} & & \\
\Downarrow & & \Downarrow & & \boxed{\begin{array}{l} \mathrm{rows}_k(A) \preccurlyeq \mathrm{rows}_k(B)\ \forall k \\ \mathrm{cols}_\ell(A) \preccurlyeq \mathrm{cols}_\ell(B)\ \forall \ell \\ \mathrm{rects}_{k,\ell}(A) \leq \mathrm{rects}_{k,\ell}(B)\ \forall k, \ell \end{array}} \\
\boxed{s_A - s_B \text{ is } F\text{-positive}} & \Rightarrow & \boxed{\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B)} & \Rightarrow &
\end{array}
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\end{array}$$

Conjecture. The rightmost implication is if and only if.

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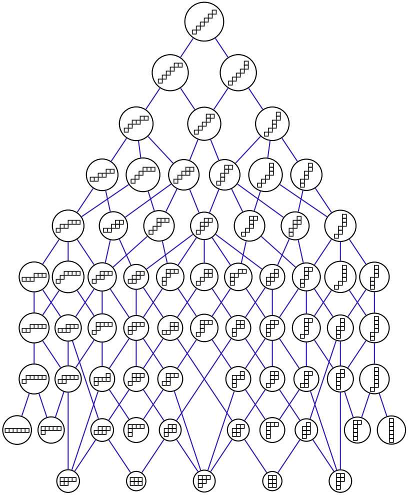
\end{array}$$

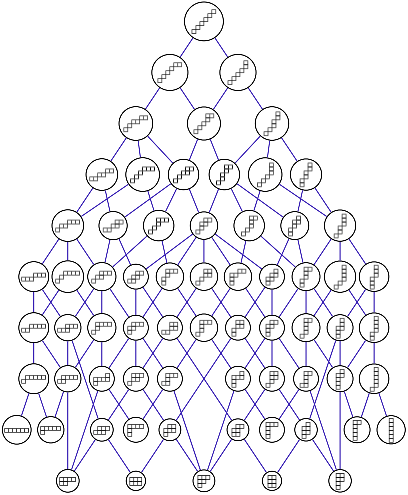
Conjecture. The rightmost implication is if and only if.

Evidence. Conjecture is true for:

- $n \leq 13$;
- horizontal strips;
- F-multiplicity-free skew shapes (as determined by Christine Bessenrodt and Steph van Willigenburg (2013));
- ribbons whose rows all have length at least 2.



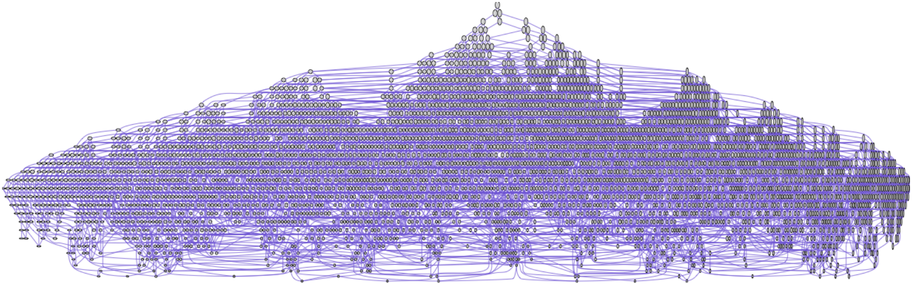
F-support containment



Dual of row overlap dominance

$n = 12$

$n = 12$ case has 12,042 edges.



$n = 13$ case has 23,816 edges.

Conclusion

$$\begin{array}{ccccc}
\boxed{s_A - s_B \text{ is Schur-pos.}} & \Rightarrow & \boxed{\operatorname{supp}_s(A) \supseteq \operatorname{supp}_s(B)} & & \\
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\boxed{s_A - s_B \text{ is } F\text{-positive}} & \Rightarrow & \boxed{\operatorname{supp}_F(A) \supseteq \operatorname{supp}_F(B)} & \overset{?}{\Leftarrow} \atop \Rightarrow & \boxed{\begin{array}{l} \operatorname{rows}_k(A) \preccurlyeq \operatorname{rows}_k(B)\ \forall k \\ \operatorname{cols}_\ell(A) \preccurlyeq \operatorname{cols}_\ell(B)\ \forall \ell \\ \operatorname{rects}_{k,\ell}(A) \leq \operatorname{rects}_{k,\ell}(B)\ \forall k, \ell \end{array}}
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\end{array}$$

Thanks! **Obrigado!**

Extras

$$
\begin{array}{ccccc}
\boxed{s_A - s_B \text{ is } D\text{-positive}} & & & & \\
\Downarrow & & \boxed{\begin{array}{c} \mathrm{supp}_D(A) \supseteq \mathrm{supp}_D(B) \\ \mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B) \\ \mathrm{supp}_S(A) \supseteq \mathrm{supp}_S(B) \end{array}} & & \\
\boxed{\begin{array}{l} s_A - s_B \text{ is Schur-pos.} \\ s_A - s_B \text{ is } S\text{-positive} \end{array}} & \Rightarrow & & & \\
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\Downarrow & & \Downarrow & & \\
\boxed{s_A - s_B \text{ is } M\text{-positive}} & \Rightarrow & \boxed{\mathrm{supp}_M(A) \supseteq \mathrm{supp}_M(B)} & &
\end{array}
$$

$$\begin{tikzcd}
\boxed{s_A - s_B \text{ is } D\text{-positive}} \arrow[d, Rightarrow] & & \\
\boxed{\begin{array}{l} s_A - s_B \text{ is Schur-pos.} \\ s_A - s_B \text{ is } S\text{-positive} \end{array}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \boxed{\begin{array}{c} \mathrm{supp}_D(A) \supseteq \mathrm{supp}_D(B) \\ \mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B) \\ \mathrm{supp}_S(A) \supseteq \mathrm{supp}_S(B) \end{array}} \arrow[d, Rightarrow] & \\
\boxed{s_A - s_B \text{ is } F\text{-positive}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \boxed{\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B)} \arrow[r, Rightarrow, shift right] \arrow[d, Rightarrow] & \boxed{\begin{array}{l} \mathrm{rows}_k(A) \preccurlyeq \mathrm{rows}_k(B)\ \forall k \\ \mathrm{cols}_\ell(A) \preccurlyeq \mathrm{cols}_\ell(B)\ \forall \ell \\ \mathrm{rects}_{k,\ell}(A) \leq \mathrm{rects}_{k,\ell}(B)\ \forall k, \ell \end{array}} \arrow[l, Rightarrow, shift right, "?"'] \\
\boxed{s_A - s_B \text{ is } M\text{-positive}} \arrow[r, Rightarrow] & \boxed{\mathrm{supp}_M(A) \supseteq \mathrm{supp}_M(B)} &
\end{tikzcd}$$

Conjecture [McN., Alejandro Morales].

A quasisym skew Saturation Theorem:

$$\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B) \iff \mathrm{supp}_F(nA) \supseteq \mathrm{supp}_F(nB).$$

Extras

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