An Introduction to the Combinatorics of Symmetric Functions

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Slides and papers available from
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What is algebraic combinatorics anyhow?

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Define combinatorics

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The biggest open problem in algebraic combinatorics:

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Combinatorics that takes its problems, or its tools, from commutative algebra, algebraic geometry, algebraic topology, representation theory, etc.

Outline

- Symmetric functions
- Schur functions and Littlewood-Richardson coefficients
- The Littlewood-Richardson rule
- Cylindric skew Schur functions

What are symmetric functions?

Definition

A symmetric polynomial is a polynomial that is invariant under any permutation of its variables $x_1, x_2, \ldots x_n$.

Examples

- $x_1 + x_2 + \cdots + x_n$
- $x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2$
  is a symmetric polynomial in x_1, x_2, x_3.

Definition

A symmetric function is a formal power series that is invariant under any permutation of its (infinite set of) variables $x = (x_1, x_2, \ldots)$.

Examples

- $\sum_{i\geq 1} x_i$ is a symmetric function, as is $\sum_{i\neq j} x_i^2x_j$.
- $\sum_{i<j} x_i^2x_j$ is not symmetric.

A basis for the symmetric functions

Fact: The symmetric functions form a vector space.
What is a possible basis?

Monomial symmetric functions: Start with a monomial:

$$x_1^7 x_2^4$$

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$$x_1^7 x_2^4 + x_1^4 x_2^7 + x_1^7 x_3^4 + x_1^4 x_3^7 + \cdots .$$

Given a *partition* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, e.g. $\lambda = (7, 4)$,

$$m_\lambda = \sum_{\substack{i_1, \ldots, i_\ell \\ \text{distinct}}} x_{i_1}^{\lambda_1} \ldots x_{i_\ell}^{\lambda_\ell}.$$

Examples

- $m_{(3)} = x_1^3 + x_2^3 + \cdots .$
- $m_{(1,1,1)}(x_1, x_2, x_3) \equiv m_{111}(x_1, x_2, x_3) = x_1 x_2 x_3.$

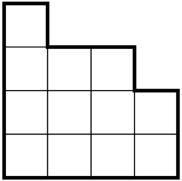
Other bases

- Elementary symmetric functions, e_λ.
- Complete homogeneous symmetric functions, h_λ.
- Power sum symmetric functions, p_λ.

Typical questions: Prove they are bases, convert from one to another, ...

Cauchy, 1815.

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.
- Young diagram.
  Example: $\lambda = (4, 4, 3, 1)$.



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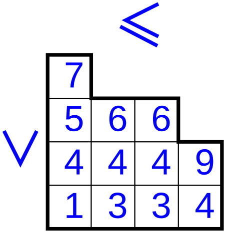
- Semistandard Young tableau (SSYT)



The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

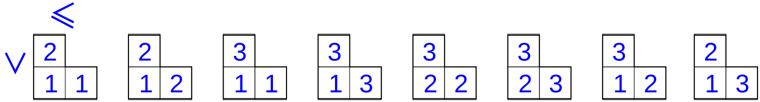
$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \cdots .$$

Example

$s_{4431} = x_1^1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \cdots .$

Schur functions

Example



Hence

$$\begin{aligned} s_{21}(x_1,x_2,x_3) &= x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \\ &\quad +2x_1x_2x_3 \\ &= m_{21}(x_1,x_2,x_3) + 2m_{111}(x_1,x_2,x_3). \end{aligned}$$

Fact: Schur functions are symmetric functions.

Question

Why do we care about Schur functions?

Why do we care about Schur functions?

- Fact: The Schur functions form a basis for the symmetric functions.
- In fact, they form an orthonormal basis: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.
- Main reason: they arise in many other areas of mathematics.
  - Representation theory of $\mathcal{S}_n$.
  - Representations of $\mathrm{GL}(n, \mathbb{C})$.
  - Algebraic Geometry: Schubert Calculus.
  - Linear Algebra: eigenvalues of Hermitian matrices.

Littlewood-Richardson coefficients

Note: The symmetric functions form a ring.

$$(x_1^2 + x_2^2 + x_3^2 + \cdots)(x_1 + x_2 + x_3 + \cdots).$$

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda.$$

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$c_{\mu\nu}^\lambda$: *Littlewood-Richardson coefficients*

Examples

- $s_{21}s_{21} = s_{42} + s_{411} + s_{33} + 2s_{321} + s_{3111} + s_{222} + s_{2211}$.
- $s_{32}s_{421} = s_{44211} + s_{54111} + s_{4332} + s_{4422} + 2s_{4431} + 2s_{5322} + 2s_{5331} + 3s_{5421} + s_{52221} + s_{5511} + s_{62211} + s_{6222} + s_{43221} + 3s_{6321} + s_{43311} + 2s_{6411} + 2s_{53211} + s_{63111} + s_{444} + 2s_{543} + s_{552} + s_{633} + 2s_{642} + s_{732} + s_{741} + s_{7221} + s_{7311} + s_{651}$.
- $c_{(8,7,6,5,4,3,2,1),(8,7,6,6,5,4,3,2,1)}^{(12,11,10,9,8,7,6,5,4,3,2,1)} = 7869992.$

  (Maple packages: John Stembridge, Anders Buch.)

Littlewood-Richardson coefficients are non-negative

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda.$$

Theorem

For any partitions μ, ν and λ,

$$c_{\mu\nu}^\lambda \geq 0.$$ *(Your take-home fact!)*

Terminology: We say that $s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda$ is a Schur-positive function.

Littlewood-Richardson coefficients are non-negative

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda.$$

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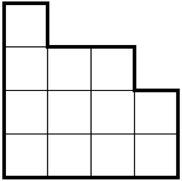
Proof 1: Use representation theory of $\mathcal{S}_n$.
Proof 2: Use representation theory of $\mathrm{GL}(n, \mathbb{C})$.
Proof 3: Use Schubert Calculus.

Want a combinatorial proof:
"They must count something simpler!"

Skew Schur functions: a generalization of Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.
- Young diagram.
  Example:
  $\lambda = (4, 4, 3, 1)$



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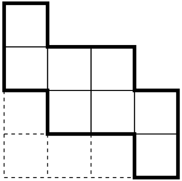
  Example:
  $\lambda/\mu = (4,4,3,1)/(3,1)$



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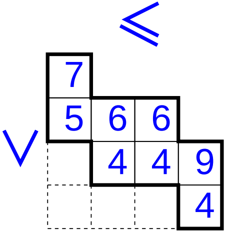

The skew Schur function $s_{\lambda/\mu}$ is the variables $x = (x_1, x_2, \ldots)$ is then defined by

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$s_{4431/31} = x_4^3 x_5 x_6^2 x_7 x_9 + \cdots$. Again, it's a symmetric function.
Remarkable fact:

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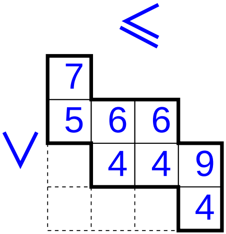

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Remarkable fact:

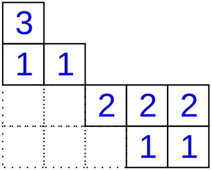
$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_\nu .$$

Littlewood-Richardson 1934, Schützenberger 1977, Thomas 1974.

Theorem

$c^{\lambda}_{\mu\nu}$ equals the number of SSYT of shape λ/μ and content ν whose reverse reading word is a ballot sequence.

Example $\lambda = (5, 5, 2, 1), \mu = (3, 2), \nu = (4, 3, 1)$



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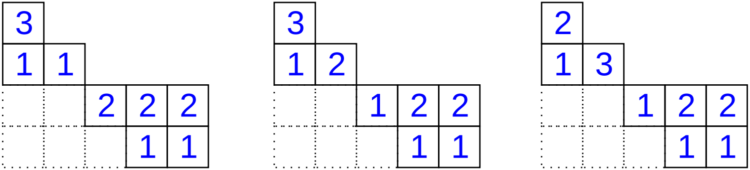

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$c^{5221}_{32,431} = 2.$

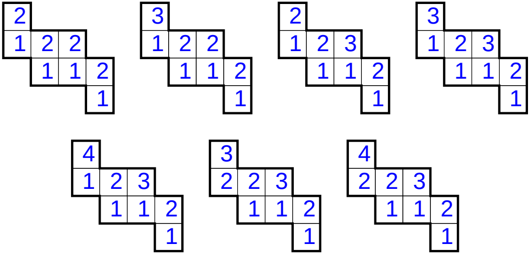
Expanding a skew Schur function

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}.$$

Can expand $s_{\lambda/\mu}$ by looking for *all* fillings of λ/μ whose reverse reading word is a ballot sequence.

Example

$\lambda/\mu = 4431/31$.



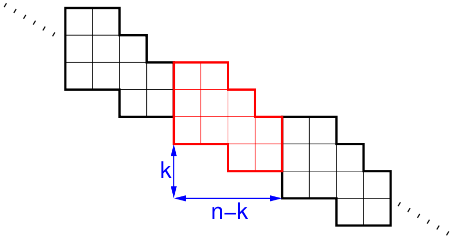
$s_{4431/31} = s_{44} + 2s_{431} + s_{422} + s_{4211} + s_{332} + s_{3311}$.

The story so far

- Schur functions: (most?) important basis for the symmetric functions
- Skew Schur functions are Schur-positive
- The coefficients in the expansion are the Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$
- The Littlewood-Richardson rule gives a combinatorial rule for calculating $c^{\lambda}_{\mu\nu}$, and hence much information about the other interpretations of $c^{\lambda}_{\mu\nu}$.

Cylindric skew Schur functions

- Infinite skew shape C
- Invariant under translation
- Identify (a, b) and $(a + n - k, b - k)$.



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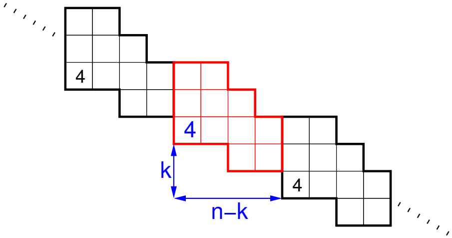

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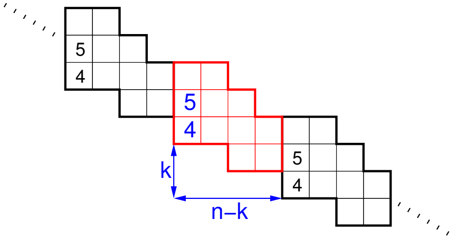

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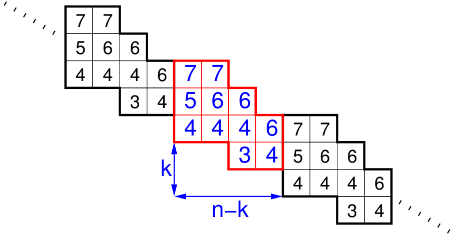

- Entries weakly increase in each row
  Strictly increase up each column
- Alternatively: SSYT with relations between entries in first and last columns
- Cylindric skew Schur function:

$$s_C(x) = \sum_T x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

$$\text{e.g. } s_C(x) = x_3 x_4^4 x_5 x_6^3 x_7^2 + \cdots .$$

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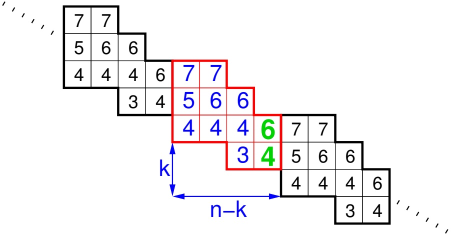

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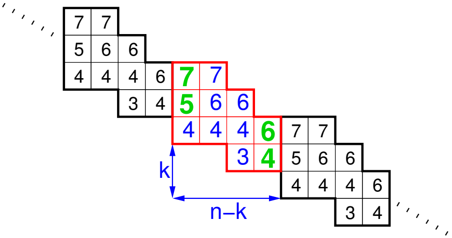

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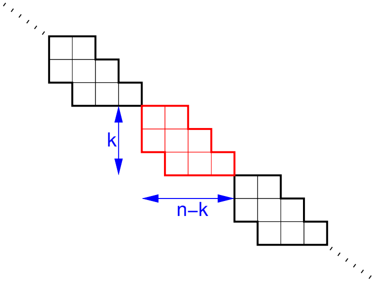
Skew shapes are cylindric skew shapes...

... and so skew Schur functions are cylindric skew Schur functions.

Example



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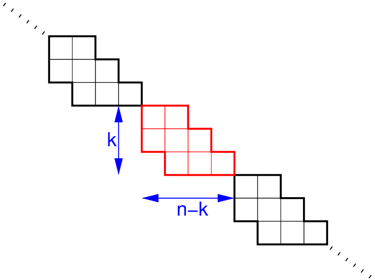

- Gessel, Krattenthaler: *"Cylindric partitions,"* 1997.
- Bertram, Ciocan-Fontanine, Fulton: *"Quantum multiplication of Schur polynomials,"* 1999.
- Postnikov: *"Affine approach to quantum Schubert calculus,"* `math.CO/0205165`.
- Stanley: *"Recent developments in algebraic combinatorics,"* `math.CO/0211114`.

Motivation: A “fundamental” open problem

A generalization of Littlewood-Richardson coefficients:
3-point Gromov-Witten invariants $C_{\mu\nu}^{\lambda,d}$.
Fact: $C_{\mu\nu}^{\lambda,d} \geq 0$ by their geometric definition.

Fundamental open problem: Find a combinatorial proof of this fact.

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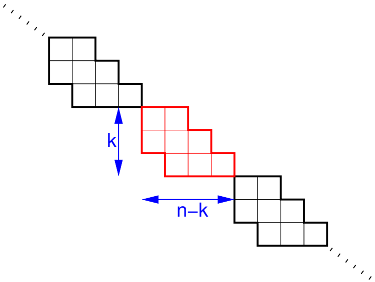
Rest of talk:

- Why the problem is difficult
- A tool
- A hint?

When is a cylindric skew Schur function Schur-positive?



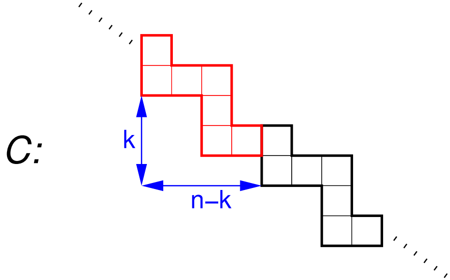
Theorem (McN.)

For any cylindric skew shape C,

$$s_C(x_1, x_2, \ldots) \text{ is Schur-positive} \Leftrightarrow C \text{ is a skew shape.}$$

Equivalently, if C is a non-trivial cylindric skew shape, then $s_C(x_1, x_2, \ldots)$ is not Schur-positive.

Example: cylindric ribbons



$$s_C(x_1, x_2, \dots) = \sum_{\lambda \subseteq k \times (n-k)} c_\lambda s_\lambda + s_{(n-k,1^k)} - s_{(n-k-1,1^{k+1})}$$
$$+ s_{(n-k-2,1^{k+2})} - \cdots + (-1)^{n-k} s_{(1^n)}.$$

Formula: cylindric skew Schur functions as signed sums of skew Schur functions

Idea for formulation: Bertram, Ciocan-Fontanine, Fulton
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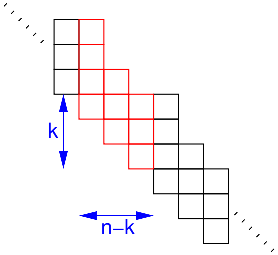

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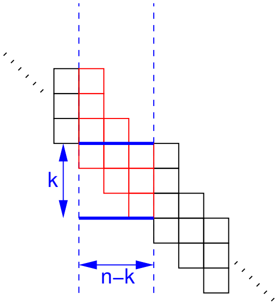

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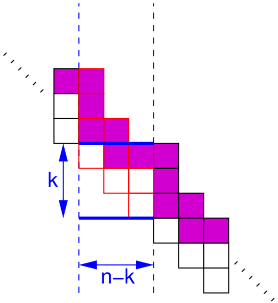

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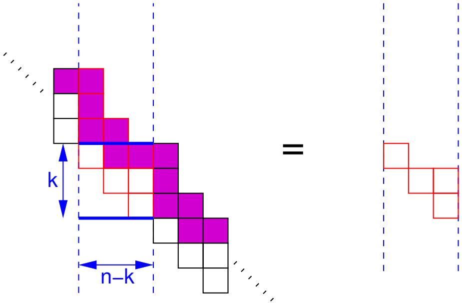

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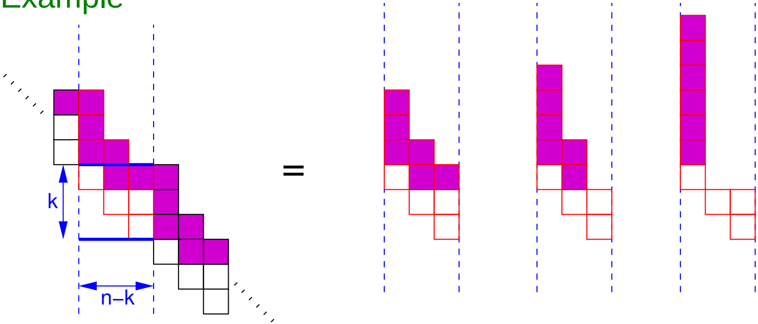

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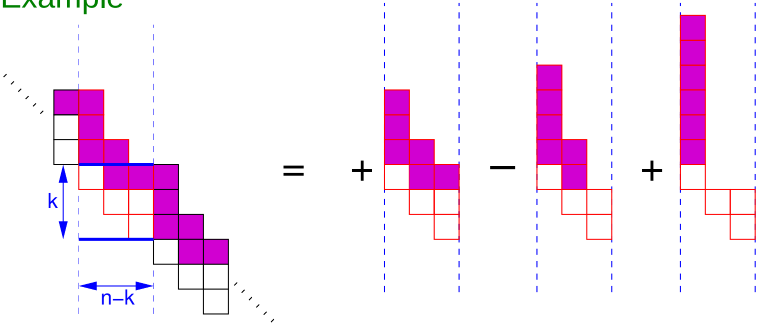

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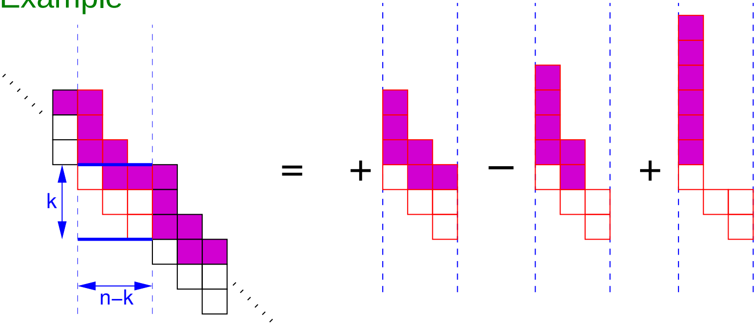
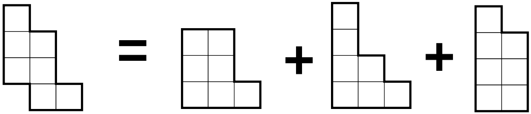


$$s_C = s_{333211/21} - s_{3322111/21} + s_{331111111/21}$$
$$= s_{3331} + s_{3322} + s_{33211} + s_{322111} + s_{31111111}$$
$$- s_{222211} - s_{2221111} + s_{22111111} + s_{211111111}.$$

A hint: Cylindric Schur-positivity

Skew Schur functions are Schur-positive:



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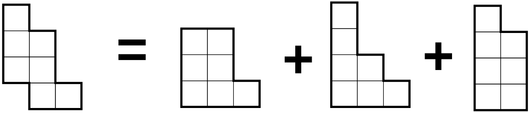
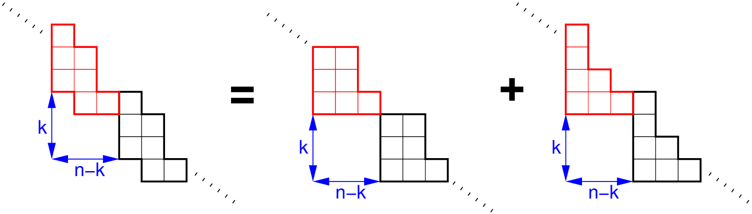


Some cylindric skew Schur functions are cylindric Schur-positive:



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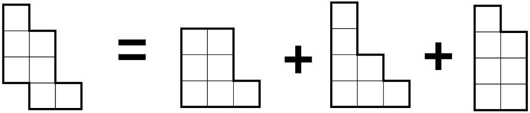

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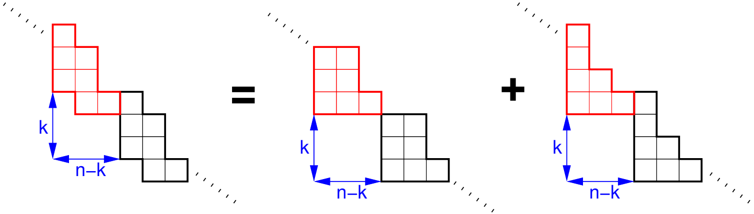
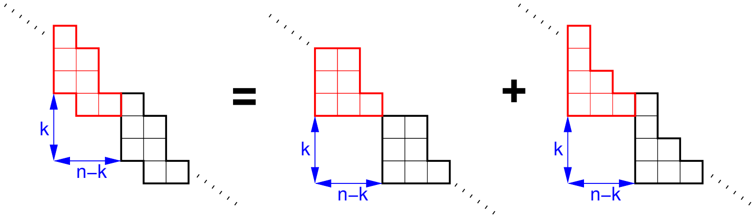


Conjecture

For any cylindric skew shape C, s_C is cylindric Schur-positive



Theorem (McN.)

The cylindric Schur functions corresponding to a given translation $(-n+k,+k)$ *are linearly independent.*

Theorem (McN.)

If s_C *can be written as a linear combination of cylindric Schur functions with the same translation as* C*, then* s_C *is cylindric Schur-positive.*

Summary of results

- Classification of those cylindric skew Schur functions that are Schur-positive.
- Full knowledge of negative terms in Schur expansion of ribbons.
- Expansion of any cylindric skew Schur function into skew Schur functions.
- Conjecture and evidence that every cylindric skew Schur function is cylindric Schur-positive.

- Outlook
  - Prove the conjecture.
  - Develop a Littlewood-Richardson rule for cylindric skew Schur functions - this would solve the “fundamental open problem.”

Another Schur-positivity research project

Know

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

is Schur-positive.

Question

Given μ, ν, when is

$$s_\sigma s_\tau - s_\mu s_\nu$$

Schur-positive? In other words, when is $c^\lambda_{\sigma\tau} - c^\lambda_{\mu\nu} \geq 0$ for every partition λ.

- Fomin, Fulton, Li, Poon: *"Eigenvalues, singular values, and Littlewood-Richardson coefficients,"* `math.AG/0301307`.
- Bergeron, Biagioli, Rosas: *"Inequalities between Littlewood-Richardson Coefficients,"* `math.CO/0403541`.
- Bergeron, McNamara: *"Some positive differences of products of Schur functions,"* `math.CO/0412289`.

Appendices

Like previous two slides, the slides that follow probably won't be included in the presentation. However, they give more details on certain aspects of what we covered.

The Schur function s_λ is a symmetric function

Proof. Consider SSYTs of shape λ and *content* $\alpha = (\alpha_1, \alpha_2, \ldots)$.

Show: $\#$ SSYTs shape λ, content $\alpha = \#$ SSYTs shape λ, content β, where β is any permutation of α.

Sufficient: $\beta = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, \ldots)$.

Bijection: SSYTs shape λ, content $\alpha \leftrightarrow$ SSYTs shape λ, content β.

$$\begin{array}{ccccc} i+1 & i+1 & & & \\ i & i & \underbrace{i \quad\quad i}_{r=2} & \underbrace{i+1 \quad i+1 \quad i+1 \quad i+1}_{s=4} & i+1 \\ & & & & i \end{array}$$

In each such row, convert the r i's and s $i+1$'s to s i's and r $i+1$'s:

$$\begin{array}{ccccc} i+1 & i+1 & & & \\ i & i & \underbrace{i \quad i \quad i \quad i}_{s=4} & \underbrace{i+1 \quad i+1}_{r=2} & i+1 \\ & & & & i \end{array}$$

□

s_λ and $c^\lambda_{\mu\nu}$ are superstars!

1. Representation Theory of S_n:

$$(S^\mu \otimes S^\nu) \uparrow^{S_n} = \bigoplus_\lambda c^\lambda_{\mu\nu} S^\lambda, \quad \text{so} \quad \chi^\mu \cdot \chi^\nu = \sum_\lambda c^\lambda_{\mu\nu} \chi^\lambda.$$

   We also have that $s_\lambda =$ the Frobenius characteristic of χ^λ.

2. Representations of $\mathrm{GL}(n, \mathbb{C})$:
   $s_\lambda(x_1, \ldots, x_n) =$ the character of the irreducible rep. V^λ.

$$V^\mu \otimes V^\nu = \bigoplus c^\lambda_{\mu\nu} V^\lambda.$$

3. Algebraic Geometry: Schubert classes σ_λ form a linear basis for $H^*(\mathrm{Gr}_{kn})$. We have

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq k \times (n-k)} c^\lambda_{\mu\nu} \sigma_\lambda.$$

   Thus $c^\lambda_{\mu\nu}$ = number of points of Gr_{kn} in $\tilde{\Omega}_\mu \cap \tilde{\Omega}_\nu \cap \tilde{\Omega}_{\lambda^\vee}$.

There's more!

4. Linear Algebra: When do there exist Hermitian matrices A, B and $C = A + B$ with eigenvalue sets μ, ν and λ, respectively?

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4. Linear Algebra: When do there exist Hermitian matrices A, B and $C = A + B$ with eigenvalue sets μ, ν and λ, respectively?

   When $c^{\lambda}_{\mu\nu} > 0$. (Heckman, Klyachko, Knutson, Tao.)

Motivation: Positivity of Gromov-Witten invariants

In $H^*(\mathrm{Gr}_{kn})$,

$$\sigma_\mu \sigma_\nu = \sum_\lambda c_{\mu\nu}^\lambda \sigma_\lambda.$$

In $QH^*(\mathrm{Gr}_{kn})$,

$$\sigma_\mu * \sigma_\nu = \sum_{d \geq 0} \sum_{\lambda \subseteq k \times (n-k)} q^d C_{\mu\nu}^{\lambda,d} \sigma_\lambda.$$

$C_{\mu\nu}^{\lambda,d}$ = 3-point Gromov-Witten invariants
= $\#\{$rational curves of degree d in Gr_{kn} that meet $\tilde{\Omega}_\mu$, $\tilde{\Omega}_\nu$ and $\tilde{\Omega}_{\lambda^\vee}\}$.

Example

$$C_{\mu,\nu}^{\lambda,0} = c_{\mu\nu}^\lambda.$$

Key point: $C_{\mu\nu}^{\lambda,d} \geq 0$.

In $H^*(\mathrm{Gr}_{kn})$,

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Example

$$C_{\mu,\nu}^{\lambda,0} = c_{\mu\nu}^{\lambda}.$$

Key point: $C_{\mu\nu}^{\lambda,d} \geq 0$.
"Fundamental open problem": Find an algebraic or combinatorial proof of this fact.

Theorem (Postnikov)

$$s_{\mu/d/\nu}(x_1, \ldots, x_k) = \sum_{\lambda \subseteq k \times (n-k)} C_{\mu\nu}^{\lambda,d} s_\lambda(x_1, \ldots, x_k).$$

Conclusion: Want to understand the expansions of cylindric skew Schur functions into Schur functions.

Connection with cylindric skew Schur functions

Theorem (Postnikov)

$$s_{\mu/d/\nu}(x_1, \ldots, x_k) = \sum_{\lambda \subseteq k \times (n-k)} C_{\mu\nu}^{\lambda,d} s_\lambda(x_1, \ldots, x_k).$$

Conclusion: Want to understand the expansions of cylindric skew Schur functions into Schur functions.

Corollary

$s_{\mu/d/\nu}(x_1, \ldots, x_k)$ is Schur-positive.

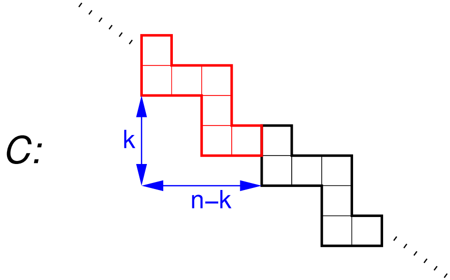
Known: $s_{\mu/d/\nu}(x_1, x_2, \ldots) \equiv s_{\mu/d/\nu}(x)$ need not be Schur-positive.

Example

If $s_{\mu/d/\nu} = s_{22} + s_{211} - s_{1111}$, then $s_{\mu/d/\nu}(x_1, x_2, x_3)$ is Schur-positive.

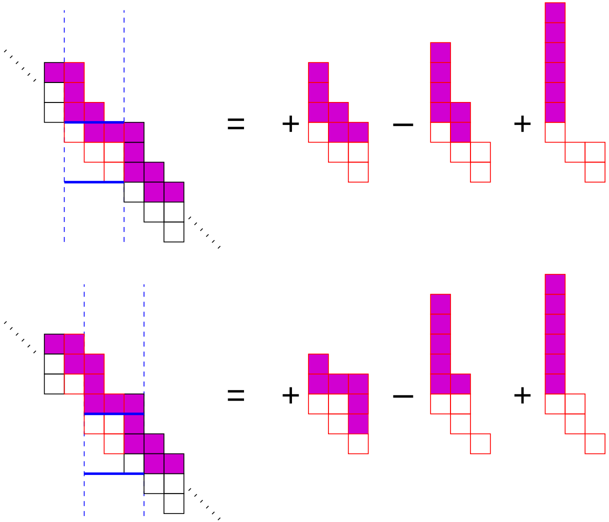
(In general: $s_\lambda(x_1, \ldots, x_k) \neq 0 \Leftrightarrow \lambda$ has at most k parts.)

Example: cylindric ribbons

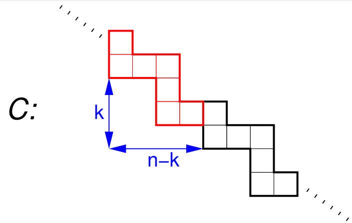


$$s_C(x_1, x_2, \ldots) = \sum_{\lambda \subseteq k \times (n-k)} c_\lambda s_\lambda + s_{(n-k,1^k)} - s_{(n-k-1,1^{k+1})}$$
$$+ s_{(n-k-2,1^{k+2})} - \cdots + (-1)^{n-k} s_{(1^n)}.$$

First consequence: lots of skew Schur function identities



A final thought: shouldn't cylindric skew Schur functions be Schur-positive *in some sense*?



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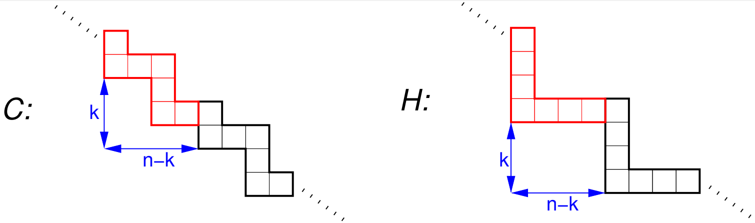

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In fact,

$$s_C(x_1, x_2, \dots) = \sum_{\lambda \subseteq k \times (n-k)} c_\lambda s_\lambda + s_H.$$

s_C: cylindric *skew* Schur function
s_H: cylindric Schur function
We say that s_C is cylindric Schur-positive.