Edge Labellings of Partially Ordered Sets and Their Implications

Peter McNamara

joint work with
Hugh Thomas

Le Séminaire de Combinatoire et d'Informatique Théorique du LaCIM
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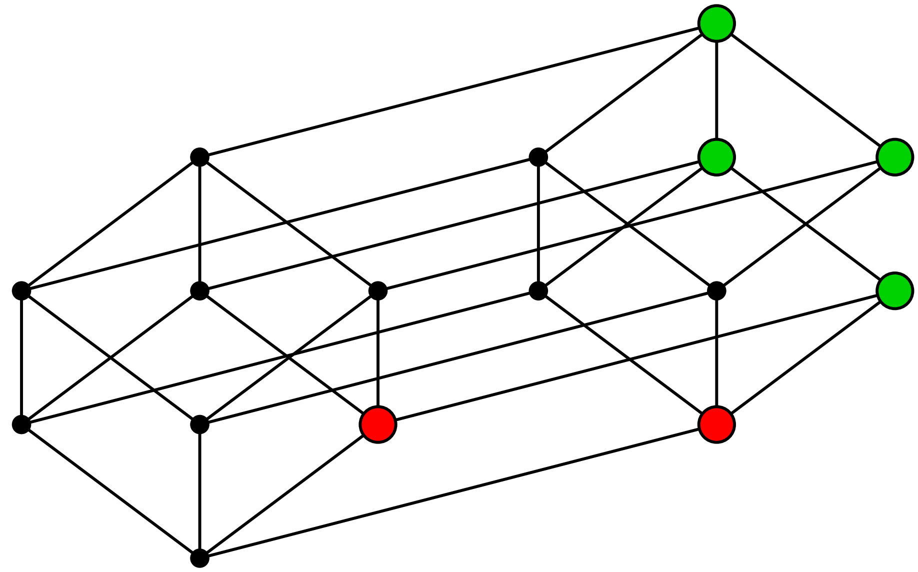
Slides and papers available from

`http://www.lacim.uqam.ca/~mcnamara/`



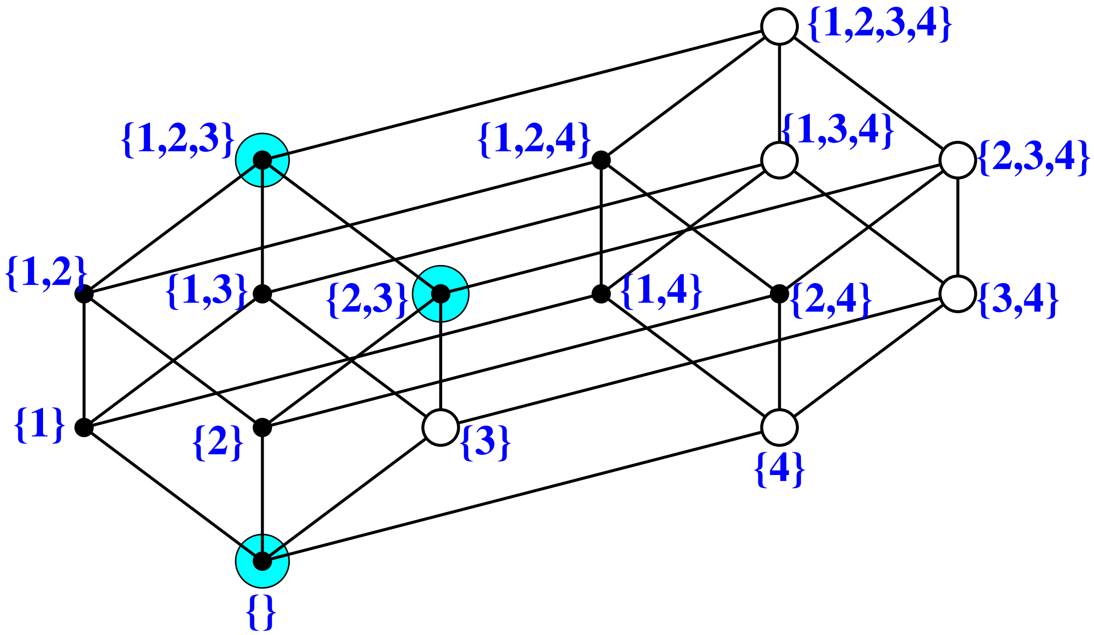
Definition P: partially ordered set (poset)
x, y: elements of P

If x and y have a least upper bound, then we call it the *join* x and y and denote it by $x \vee y$.

If x and y have a greatest lower bound, then we call it the *meet* of x and y and denote it by $x \wedge y$.

A *lattice* is a poset in which every two elements have a meet and a join.

(All our posets will be finite.)


{1,2,3,4}
{1,2,3}
{1,2,4}
{1,3,4}
{2,3,4}
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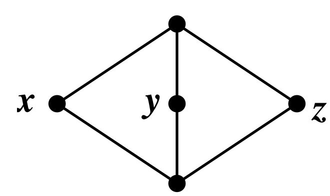
{2,4}
{3,4}
{1}
{2}
{3}
{4}
{}

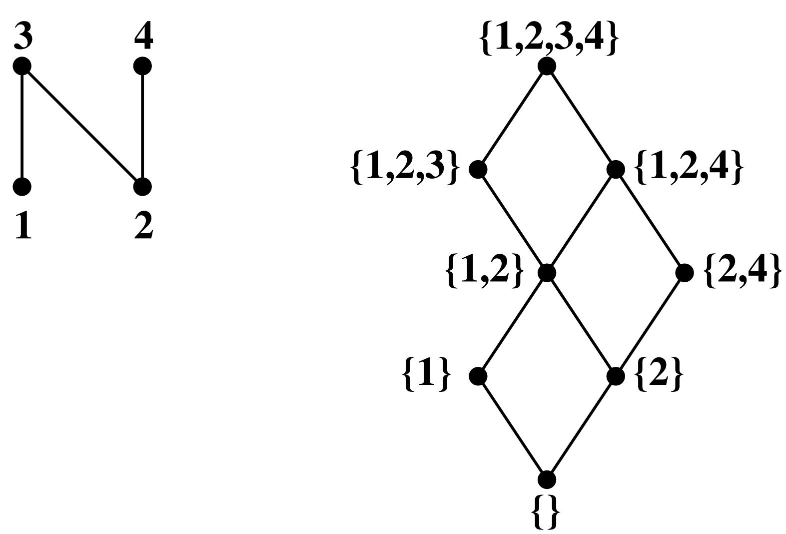
We say that a lattice L is *distributive* if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \text{and}$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

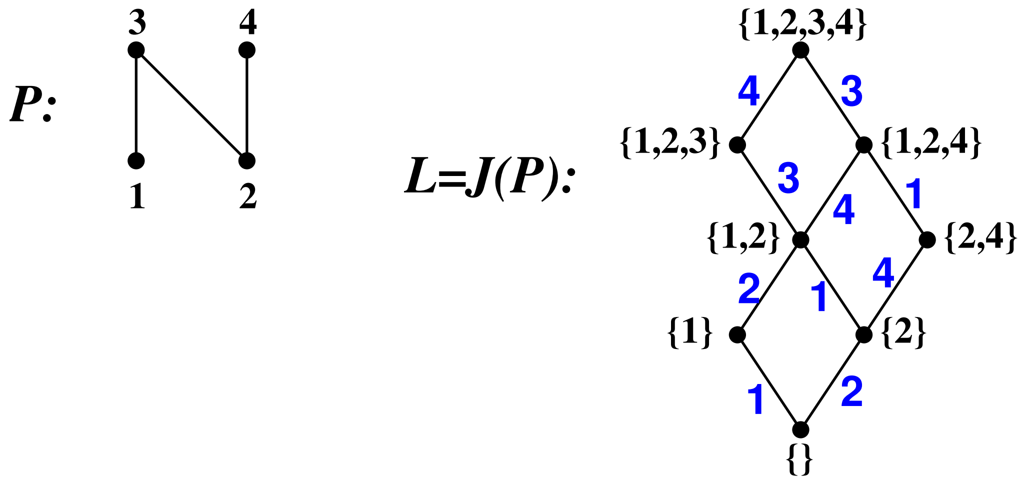
for all elements x, y and z of L.



EXAMPLE An *order ideal* of a poset P is a subset I of P such that if $x \in I$ and $y \leq x$, then $y \in I$. The lattice of order ideals of a poset P is a distributive lattice.

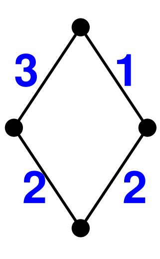


THEOREM*(FTFDL Birkhoff) A finite lattice L is distributive if and only if it is the lattice $J(P)$ of order ideals of some poset P.*

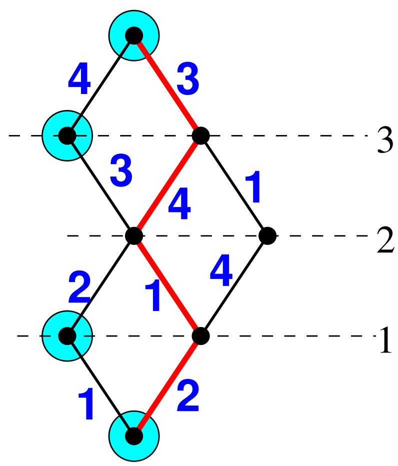


Definition An edge labelling of a poset P is said to be an *EL-labelling* if:

1. Every interval $[x, y]$ of P has exactly one maximal chain with increasing labels
2. The sequence of labels along this increasing maximal chain lexicographically precede the labels along any other maximal chain of $[x, y]$.



Who cares? P is a bounded graded poset of rank n. Let S be any subset of $[n-1]$.



- Flag f-vector $\alpha_P(S)$: number of chains in P with rank set S.

  If P has an EL-labelling: number of maximal chains of P with descent set contained in S.

- Flag h-vector $\beta_P(S)$:

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T).$$

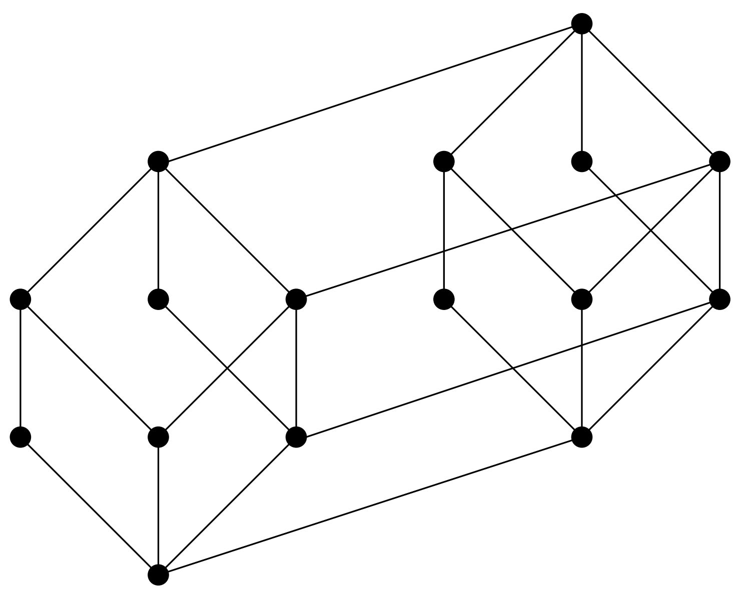
  If P has an EL-labelling: number of maximal chains of P with descent set S. So $\beta_P(S) \geq 0$.

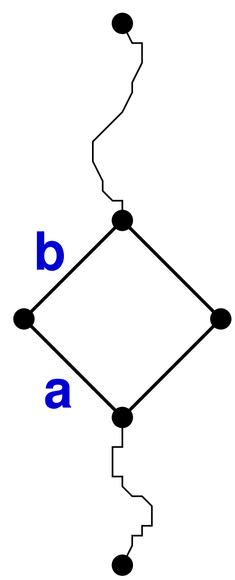
- Möbius function: $\mu(\hat{0}, \hat{1}) = (-1)^n \beta_P([n-1])$.

- EL-labelling $\Rightarrow$ Shellable $\Rightarrow$ Cohen-Macaulay

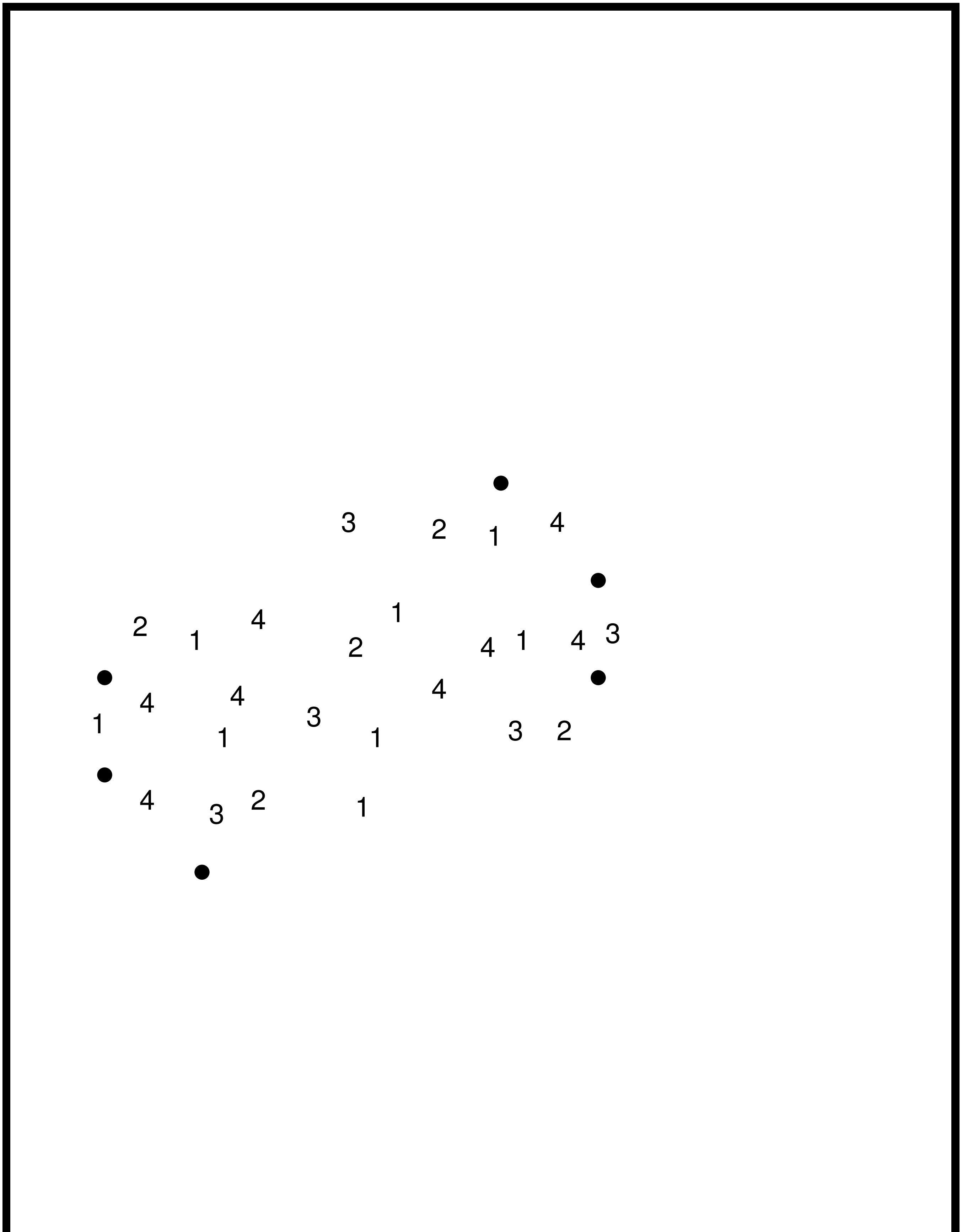
Definition An edge labelling of a poset P is said to be an *S_n EL-labelling* if:

1. Every interval $[x, y]$ of P has exactly one maximal chain with increasing labels
2. The labels along any maximal chain form a permutation of $[n]$.



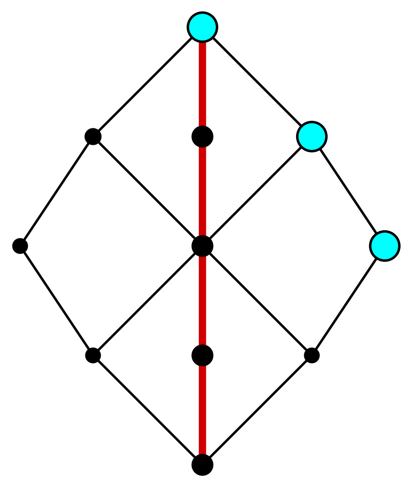


What other classes of posets have S_n EL-labellings?



Definition (R. Stanley, '72) A finite lattice L is said to be *supersolvable* if it contains a maximal chain $\mathfrak{m}$, called an *M-chain* of L which together with any other chain of L generates a distributive sublattice.

EXAMPLES



- Distributive lattices
- Modular lattices
- The lattice of partitions of $[n]$
- The lattice of non-crossing partitions of $[n]$
- The lattice of subgroups of a supersolvable group

QUESTION (Stanley) *"Are there any other lattices that have S_n EL-labellings?"*

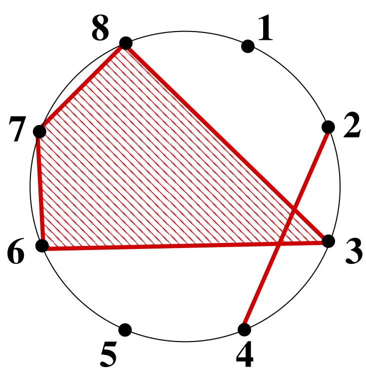
THEOREM (McN.) *A lattice is supersolvable if and only if it has an S_n EL-labelling.*

EXAMPLE Biagioli & Chapoton: Lattice of leaf labelled binary trees
`www.arxiv.org/math.CO/0304132`

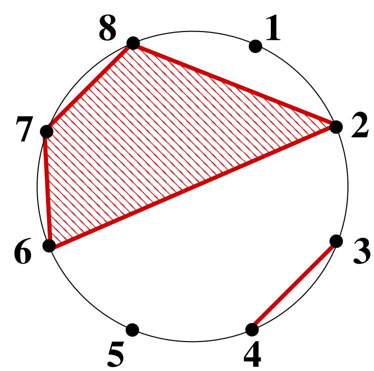
Example A partition of $[n]$ into unordered blocks is said to be *non-crossing* if

$$i < j < k < l \quad \text{with} \quad i, k \in B \quad \text{and} \quad j, l \in B'$$

$$\text{implies } B = B'.$$

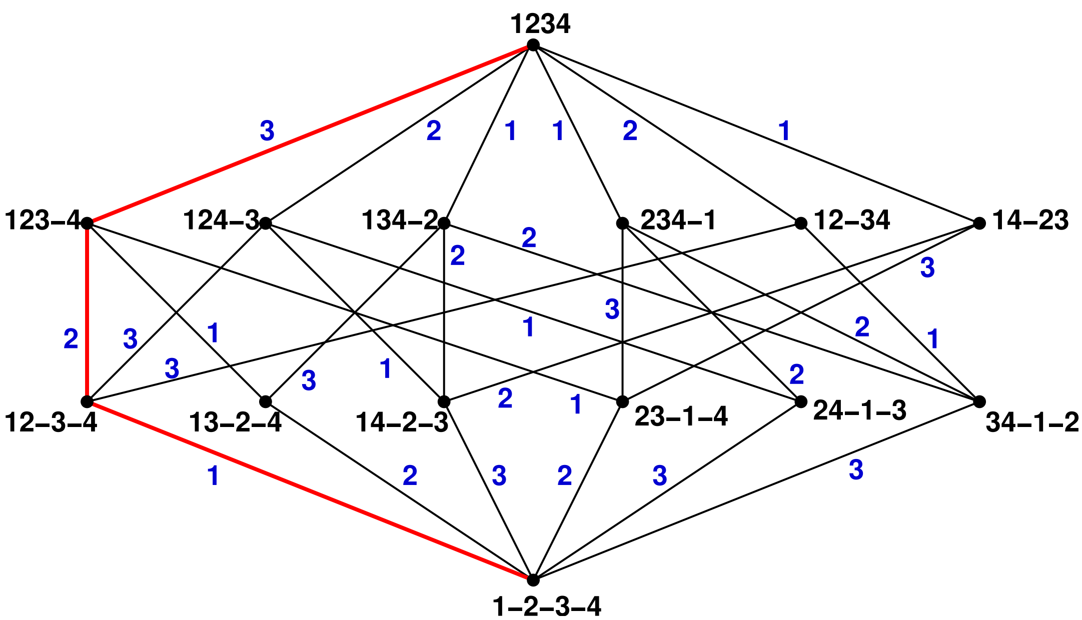


1–24–3678–5
crossing



1–2678–34–5
non–crossing

S_n EL-labelling: Björner and Edelman



Connections with modularity...

Suppose L is lattice with $y \le z$. Always true:

$$(x \vee y) \wedge z \qquad (x \wedge z) \vee y.$$

Definition An element x of a lattice L is said to be *left modular* if, for all $y \le z$ in L, we have

$$(x \vee y) \wedge z = (x \wedge z) \vee y.$$

A chain of L is *left modular* if each of its elements is left modular.

Suppose L is a **graded** lattice.

L is supersolvable	$\Longleftrightarrow$	L has an S_n EL-labelling

L has a left modular maximal chain

Stanley ⇘

$\Nearrow$ Liu

THEOREM *Let L be graded lattice. TFAE:*

1. *L is supersolvable*
2. *L has an S_n EL-labelling*
3. *L has a left modular maximal chain*
4.

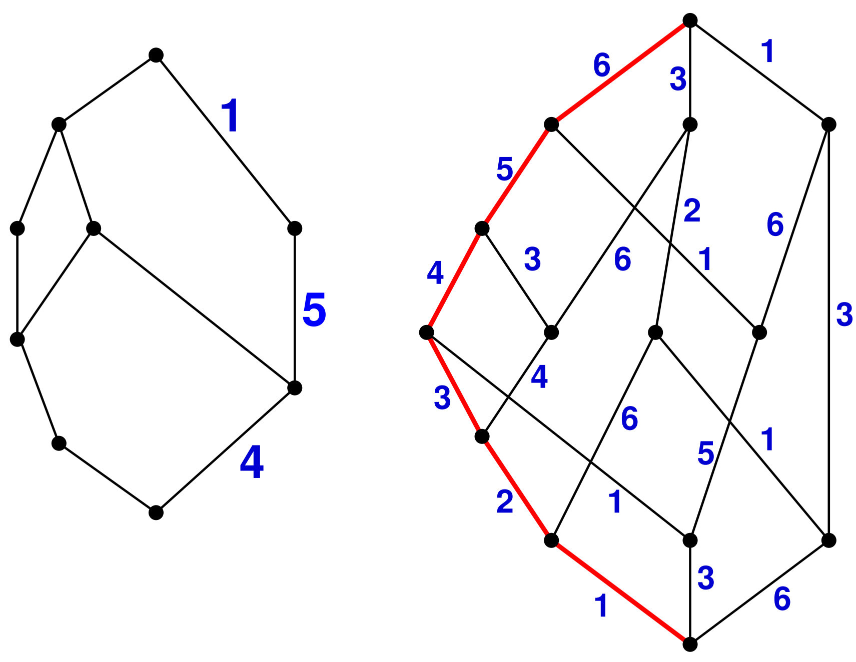
How can we extend this?

- 3: L need not be graded
- 2: L need not be a lattice

Definition Let P be a bounded poset. An EL-labelling γ of P is said to be *interpolating* if, for any $y \lessdot u \lessdot z$, either

(i) $\gamma(y,u) < \gamma(u,z)$ or

(ii) the increasing chain from y to z , say $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$, has the properties that its labels are strictly increasing and that $\gamma(w_0,w_1) = \gamma(u,z)$ and $\gamma(w_{r-1},w_r) = \gamma(y,u)$.

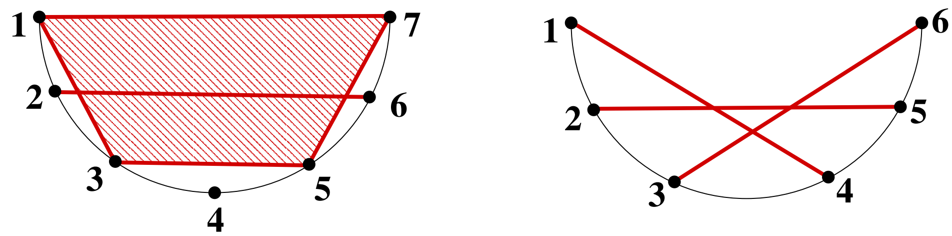


THEOREM (Thomas) *A lattice has an interpolating EL-labelling if and only if it has a left modular maximal chain.*

Example A partition of $[n]$ into unordered blocks is said to be if

$$i < j < k < l \quad \text{with} \qquad \in B \quad \text{and} \qquad \in B'$$

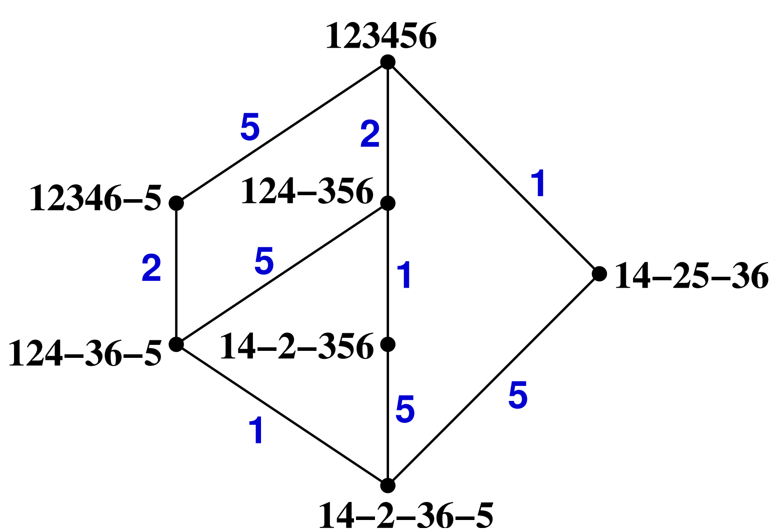
$$\text{implies} \quad B = B'.$$



1357–26–4
straddling

14–25–36
non–straddling

Ordering, edge labelling: Analogous to non-crossing partitions



non-crossing

i, k j, l

non-straddling

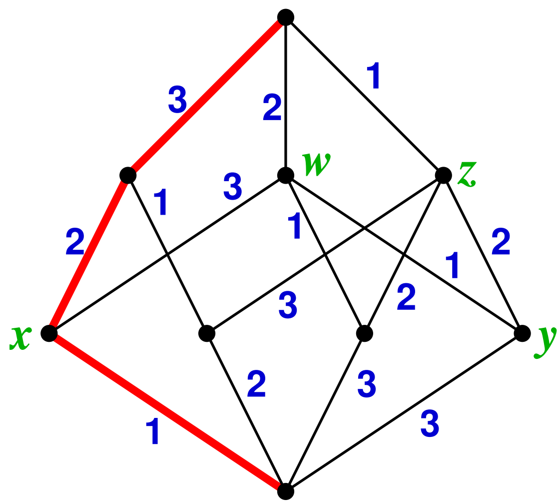
i, l j, k

Generalizing to non-lattices:

P: a bounded poset with an S_n EL-labelling.

$\mathfrak{m}$: its increasing maximal chain.

Some "left modularity" property ?



When $x \in \mathfrak{m}$, $x \vee y$ and $x \wedge y$ are well-defined.

In a lattice: $(x \vee y) \wedge z \geq y$ whenever $z \geq y$.

When $x \in \mathfrak{m}$, $(x \vee y) \wedge_y z$ is well-defined for $y \leq z$.

Similarly, $(x \wedge z) \vee^z y$ is well-defined.

We call x a *viable* element of P.

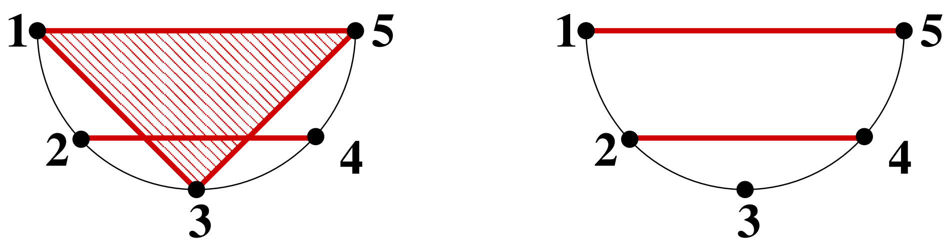
We call $\mathfrak{m}$ a *viable* maximal chain.

THEOREM (McN.-Thomas) *A bounded poset has an interpolating EL-labelling if and only if it has a viable left modular maximal chain.*

Example A partition of $[n]$ into unordered blocks is said to be if

$$i < j < k < l \quad \text{with} \quad i, l \qquad\qquad B \quad \text{and}$$

$$j, k \in B' \quad \text{implies} \quad B = B'.$$



135–24
straddling
non–nesting

15–24–3
straddling
nesting

- Ordering, edge labelling: Analogous to non-crossing partitions
- Like non-straddling partitions, poset is not graded
- Not even a lattice: Consider 136-25-4 $\wedge$ 146-25-3.

non-straddling

$\in$

non-nesting

adjacent in

Finally, generalizing supersolvability:

Suppose P has a viable maximal chain $\mathfrak{m}$. So $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ are well-defined for $x \in \mathfrak{m}$ and $y \leq z$ in P.
Given any chain $\mathfrak{c}$ of P, we define $R_{\mathfrak{m}}(\mathfrak{c})$ to be the smallest subposet of P satisfying:

(i) $\mathfrak{m}$ and $\mathfrak{c}$ are contained in $R_{\mathfrak{m}}(\mathfrak{c})$,

(ii) If $y \leq z$ in P and y and z are in $R_{\mathfrak{m}}(\mathfrak{c})$, then so are $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ for any x in $\mathfrak{m}$.

Definition We say that a finite bounded poset P is *supersolvable* with M-chain $\mathfrak{m}$ if $\mathfrak{m}$ is a viable maximal chain and $R_{\mathfrak{m}}(\mathfrak{c})$ is a distributive lattice for any chain $\mathfrak{c}$ of P.

THEOREM (McN.-Thomas) *Let P be a bounded graded poset of rank n. TFAE:*

1. *P has an S_n EL-labelling*
2. *P has a viable left modular maximal chain*
3. *P is supersolvable*

	Graded	**Not nec. graded**
Lattice	1. Supersolvable 2. S_n EL-labelling 3. Left mod. max. chain	1. ? 2. Interp. EL-labelling 3. Left mod. max. chain
Not nec. Lattice	1. Supersolvable 2. S_n EL-labelling 3. Viable left mod. m.c.	1. ? 2. Interp. EL-labelling 3. Viable left mod. m.c.

How can generalise supersolvability to the non-graded case?

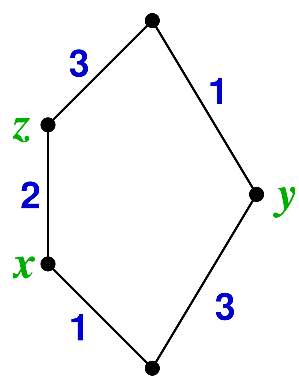
Recall:

Definition A finite lattice L is said to be *supersolvable* if it contains a maximal chain which together with any other chain of L generates a distributive sublattice.

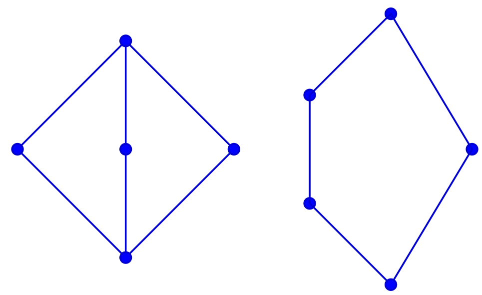
Definitions of distributive:

1. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all x, y, z.

   Problem:

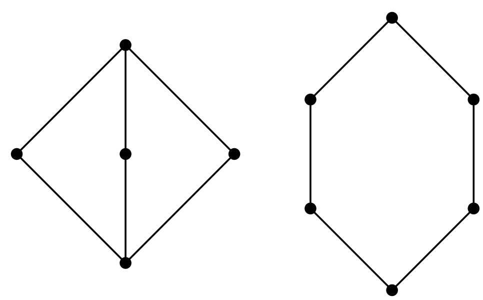


2. No sublattices of the following two forms:



What about:

A lattice is *distributive* if it has no sublattices of the following two forms:



Difficult to work with.

3. The lattice of order ideals of some poset.

What about:

A lattice is *distributive* if it is the lattice of *augmented order ideals* of some *augmented poset*.

EXAMPLE