The Structure of the Consecutive Pattern Poset

Peter McNamara
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Joint work with:
Sergi Elizalde
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and
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AMS Special Session on *Applications of Partially Ordered Sets in Enumerative, Topological, and Algebraic Combinatorics*

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Outline

- Permutation patterns: classical and consecutive
- Consecutive pattern poset
- Results
- 10 open problems

Classical patterns

Definition. An occurrence of a permutation σ as a pattern in a permutation τ is a subsequence of τ whose letters are in the same relative order as those in σ.

Example. 231 occurs in twice in 416325: 416325 and 416325.

Example. An inversion in τ is equivalent to an occurrence of 21, e.g. 1423 and 1423.

Classical patterns

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Example. An inversion in τ is equivalent to an occurrence of 21, e.g. 1423 and 1423.

- Huge area of study in the last three decades.
- Most work is enumerative, esp. counting the number of permutations that avoid a given pattern.
- Knuth (1975), Rogers (1978): For any permutation $\sigma \in S_3$, the number of permutations in S_n avoiding σ is C_n.
- Open: closed formula for number avoiding 1324.

Consecutive patterns

Our focus:

Definition. An occurrence of a consecutive pattern σ in a permutation τ is a subsequence of adjacent letters of τ in the same relative order as those in σ.

Examples.

- 123 occurs twice in 7245136: 7245136 and 7245136.
- 416325 avoids the consecutive pattern 231.
- A descent is an occurrence of the consecutive pattern 21, e.g 4132 and 4132.
- A peak is an occurrence of 132 or 231, e.g., 13415.
- A permutation is alternating (up-down or down-up) iff it avoids 123 and 321 as consecutive patterns.

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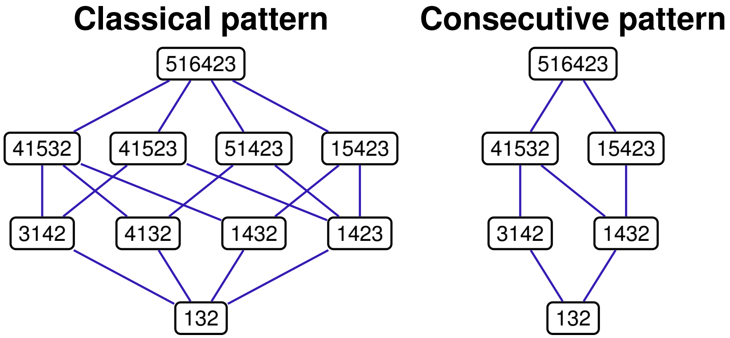
- A peak is an occurrence of 132 or 231, e.g., 13415.
- A permutation is alternating (up-down or down-up) iff it avoids 123 and 321 as consecutive patterns.
- Elizalde–Noy (2003), Aldred, Amigó, Atkinson, Bandt, Baxter, Bernini, Bóna, Dotsenko, Duane, Dwyer, Ehrenborg, Ferrari, Keller, Kennel, Khoroshkin, Kitaev, Liese, Liu, Mansour, McCaughan, Mendes, Nakamura, Perarnau, Perry, Pompe, Pudwell, Rawlings, Remmel, Sagan, Shapiro, Steingrímsson, Warlimont, Willenbring, Zeilberger, . . .

Pattern posets

Pattern order: order permutations by pattern containment.
$\sigma \leq \tau$ if σ occurs as a pattern it τ.



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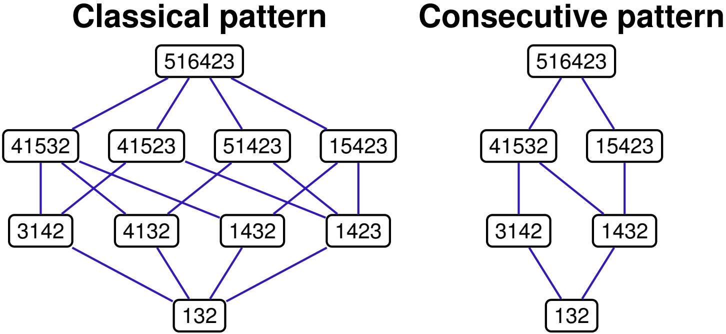
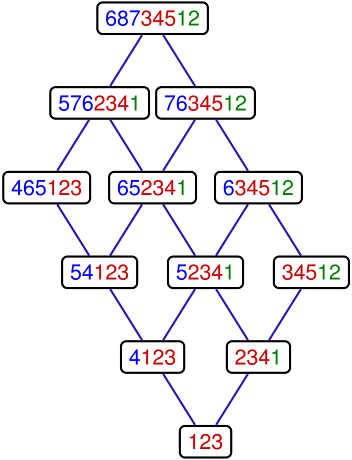


Consecutive pattern poset is more manageable.

- Consecutive case: every permutation covers at most two others.
- Wilf (2002): Möbius function $\mu(\sigma, \tau)$ of the pattern poset?
    - Known in consecutive case: Sagan–Willenbring, Bernini–Ferrari–Steingrímsson (2011).

Pattern posets

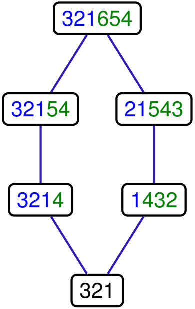
- Consecutive case: when σ occurs just once in τ, $[\sigma, \tau]$ is a product of two chains [BFS11].



Classical case: wide open even in this special case.

Main questions

Unless otherwise specified: consecutive pattern poset.



1. Which open intervals are disconnected?
2. Which intervals are rank-unimodal?
3. Which intervals are shellable?
4. Which intervals are strongly Sperner?
5. Which intervals have Möbius function equal to 0?

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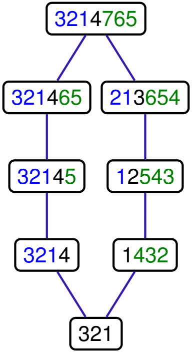

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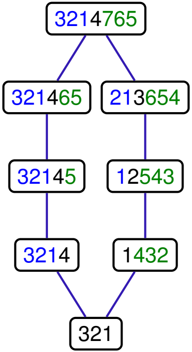
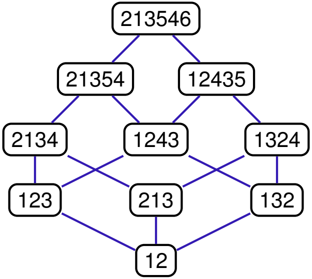


Theorem [Elizalde, McN.]. For $\sigma < \tau$ with $|\tau| - |\sigma| \geq 3$, we have that the open interval (σ, τ) is disconnected if and only if σ straddles τ. In this case, (σ, τ) consists of two disjoint chains.

2. Which intervals are rank-unimodal?



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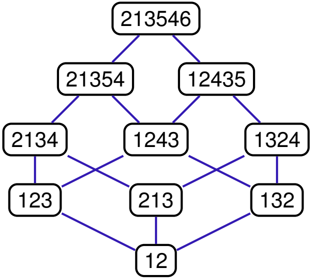

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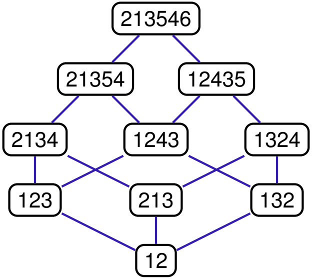

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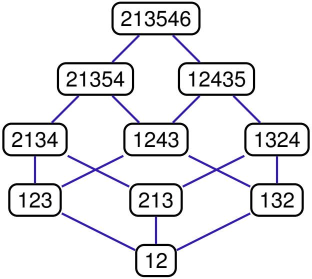

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Theorem [Elizalde & McN.]. Every interval $[\sigma, \tau]$ is rank-unimodal.

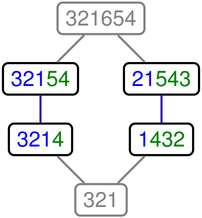
Conjecture [McN. & Steingrímsson] Every interval $[\sigma, \tau]$ in the classical pattern poset is rank-unimodal.

True for intervals of rank ≤ 8.

3. Which intervals are shellable?

Enough to know about shellability:

Main non-shellable example. If (σ, τ) is disconnected with $|\tau| - |\sigma| \geq 3$ then $[\sigma, \tau]$ is not shellable.



These intervals are said to be non-trivially disconnected.

Moreover: if $[\sigma, \tau]$ contains a non-trivial disconnected subinterval, then $[\sigma, \tau]$ is not shellable.

3. Which intervals are shellable?

Consequence: almost all intervals are not shellable.

Theorem [McN. & Steingrímsson; Elizalde & McN.].
Fix σ. Randomly choosing τ of length n,

$$\lim_{n\to\infty}(\text{Probability that } [\sigma, \tau] \text{ is shellable}) = 0.$$

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What about intervals without non-trivial disconnected subintervals?

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Corollary [Sagan & Willenbring]. Such intervals are either contractible or homotopic to a sphere.

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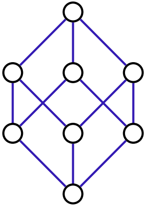
Example.

Sperner:



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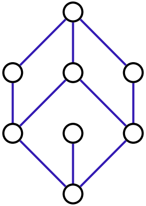

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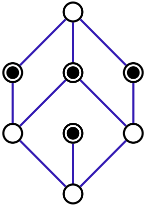

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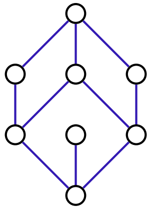

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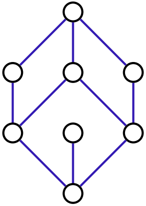

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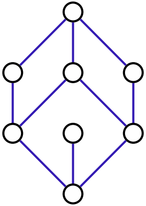

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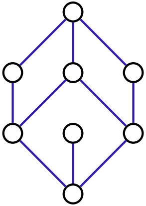

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Theorem [Elizalde & McN.]. Every interval $[\sigma, \tau]$ is strongly Sperner.

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Interior $i(\tau)$: the permutation pattern obtained by deleting first and last element of τ.
Exterior $x(\tau)$: the longest proper prefix that is also a suffix.

Examples.
$\tau = 21435$, $i(\tau) = 132$, $x(\tau) = 213$
$\tau = 123456$ (monotone), $x(\tau) = 12345$
$\tau = 654321$ (monotone), $x(\tau) = 54321$
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Theorem [SW, BFS (2011)]. For $\sigma \leq \tau$,

$$\mu(\sigma,\tau) = \begin{cases} \mu(\sigma, x(\tau)) & \text{if } |\tau| - |\sigma| > 2 \text{ and } \sigma \leq x(\tau) \not\leq i(\tau), \\ 1 & \text{if } |\tau| - |\sigma| = 2,\ \tau \text{ is not monotone,} \\ & \text{and } \sigma \in \{i(\tau), x(\tau)\}, \\ (-1)^{|\tau|-|\sigma|} & \text{if } |\tau| - |\sigma| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

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Note. $x(\tau)$ plays a crucial role.

5. Which intervals have Möbius function equal to 0?

Answer. Almost all of them.

Theorem [Elizalde & McN.]. Fix σ. Randomly choosing τ of length n with $\tau \geq \sigma$,

$$\lim_{n \to \infty} (\text{Probability that } \mu(\sigma, \tau) = 0) = 1.$$

Length of the exterior

$n\backslash k$	1	2	3	4	5	6	7	8	9
2	2								
3	4	2							
4	12	10	2						
5	48	58	12	2					
6	280	306	118	14	2				
7	1864	2186	822	150	16	2			
8	14840	17034	6580	1660	186	18	2		
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Lemma [Elizalde & McN.]. Next diagonal values when $n \geq 4$ are $2n + 2$.

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Theorem [Elizalde & McN.]. $e - 1 \ \leq \ \lim_{n\to\infty} \mathbb{E}_n(|x(\tau)|) \ \leq \ e.$

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Unknown. Everything else.

Open problems

Open problems: exterior

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2. For each k, find $\lim_{n\to\infty} \mathbb{P}_n(|x(\tau)| = k)$. (Know limit exists.)
   Bóna: $0.3640981 \leq \lim_{n\to\infty} \mathbb{P}_n(|x(\tau)| = 1) \leq 0.3640993$.
3. Find the exact value of $\lim_{n\to\infty} \mathbb{E}_n(|x(\tau)|)$ (about 1.9127).
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3	4	2							
4	12	10	2						
5	48	58	12	2					
6	280	306	118	14	2				
7	1864	2186	822	150	16	2			
8	14840	17034	6580	1660	186	18	2		
9	132276	154162	58854	15118	2222	226	20	2	
10	1323504	1532574	588898	150388	30238	2904	270	22	2

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2. For each k, find $\lim_{n\to\infty} \mathbb{P}_n(|x(\tau)| = k)$. (Know limit exists.)
   Bóna: $0.3640981 \leq \lim_{n\to\infty} \mathbb{P}_n(|x(\tau)| = 1) \leq 0.3640993$.
3. Find the exact value of $\lim_{n\to\infty} \mathbb{E}_n(|x(\tau)|)$ (about 1.9127).
4. Question [Aguair]. Is ith off-diagonal polynomial of degree i?

Open problems: exterior

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Open problems: pattern posets

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5. Characterize those intervals $[\sigma, \tau]$ that are lattices (in terms of easy conditions on σ and τ).
6. Find an easy classification of intervals that contain no non-trivial disconnected subintervals (and are thus shellable).

Classical case:

7. The question that started it all: what's the Möbius function $\mu(\sigma, \tau)$?
8. Prove the rank-unimodality conjecture.
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General:

10. Find a good way to test shellability by computer.

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Thanks!