The Möbius function of generalized subword order

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Joint work with:
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AMS Special Session on Enumerative and Algebraic Combinatorics
January 5, 2012

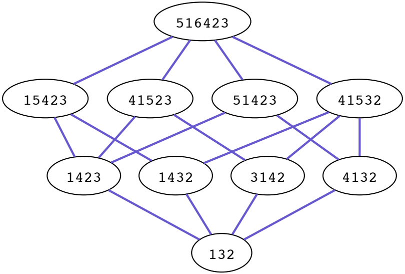
Slides and paper (*Adv. Math.*, to appear) available from
`www.facstaff.bucknell.edu/pm040/`

Outline

- Generalized subword order and related posets
- Main result
- Applications

Motivation: Wilf's question

Pattern order: order permutations by pattern containment.



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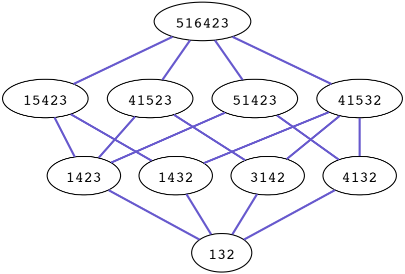

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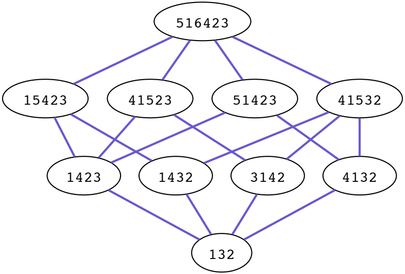

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- Sagan & Vatter (2006)
- Steingrímsson & Tenner (2010)
- Burstein, Jelínek, Jelínková & Steingrímsson (2011)

Still open.

Motivation for generalized subword order

P-partitions interpolate between partitions and compositions of n.

Generalized subword order interpolates between two partial orders.

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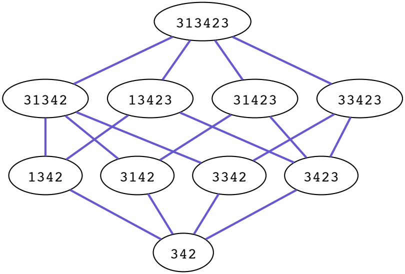
P-partitions interpolate between partitions and compositions of *n*.

Generalized subword order interpolates between two partial orders.

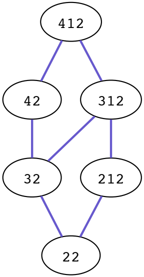
1. Subword order.
   A^*: set of finite words over alphabet *A*.
   $u \leq w$ if u is a subword of w, e.g., 342 $\leq$ 313423.



   Björner (1998): Möbius function.

Motivation for generalized subword order

2. An order on compositions.
$(a_1, a_2, \ldots, a_r) \leq (b_1, b_2, \ldots, b_s)$ if there exists a subsequence $(b_{i_1}, b_{i_2}, \ldots, b_{i_r})$ such that $a_j \leq b_{i_j}$ for $1 \leq j \leq r$.
e.g. $22 \leq 412$.



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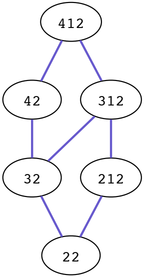

Sagan & Vatter (2006): Möbius function.

Composition order $\cong$ pattern order on *layered* permutations

$412 \leftrightarrow 4321576$

Generalized subword order

P: any poset.

P^*: set of words over the alphabet P.

Main Definition. $u \le w$ if there exists a subword $w(i_1)w(i_2)\cdots w(i_r)$ of w of the same length as u such that

$$u(j) \le_P w(i_j) \text{ for } 1 \le j \le r.$$

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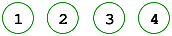
Example 1. If P is an antichain, $u(j) \le_P w(i_j)$ iff $u(j) = w(i_j)$. Gives subword order on the alphabet P, e.g., $342 \le 313423$.



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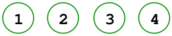
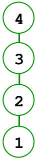
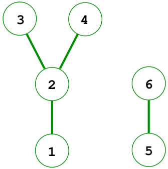




Example 2. If P is a chain, $u(j) \leq_P w(i_j)$ iff $u(j) \leq w(i_j)$ as integers. Gives composition order, e.g. $22 \leq 412$.

Definition from Sagan & Vatter (2006); appeared earlier in context of well quasi-orderings [Kruskal, 1972 survey].

Example 3. Sagan & Vatter: *P* is a *rooted forest*.



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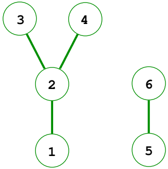
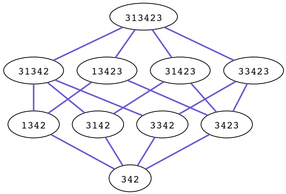
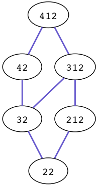


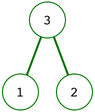
Includes antichains and chains.
Sagan & Vatter: Möbius function $\mu(u, w)$ for any $u, w \in P^*$.





Key example

Example 4. $P = \Lambda$



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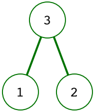
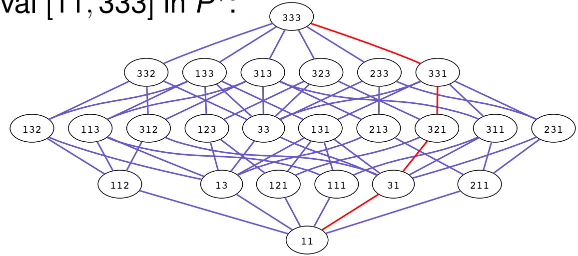


The interval $[11, 333]$ in P^*:



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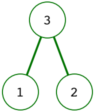

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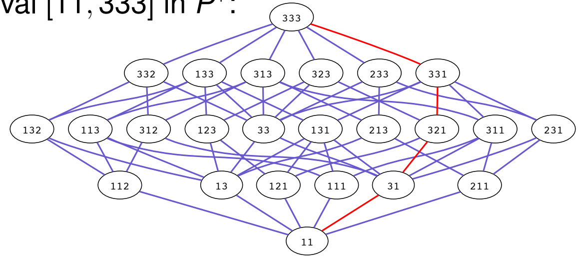

Sagan & Vatter: conjecture that $\mu(1^i, 3^j)$ equals certain coefficients of Chebyshev polynomials of the first kind.

Tomie (2010): proof using ad-hoc methods.

Our first goal: a more systematic proof.

Main result

P_0: P with a bottom element 0 adjoined.
μ_0: Möbius function of P_0.

Theorem. Let P be a poset so that P_0 is locally finite. Let u and w be elements of P^* with $u \le w$. Then

$$\mu(u, w) = \sum_{\eta} \prod_{1 \le j \le |w|} \begin{cases} \mu_0(\eta(j), w(j)) + 1 & \text{if } \eta(j) = 0 \text{ and} \\ & w(j-1) = w(j), \\ \mu_0(\eta(j), w(j)) & \text{otherwise,} \end{cases}$$

where the sum is over all embeddings η of u in w.

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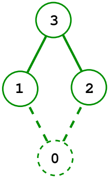
Example. Calculate $\mu(11, 333)$ when $P = \Lambda$.



w	$=$	333	333	333
η	$=$	110	101	011

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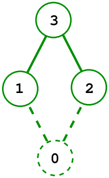

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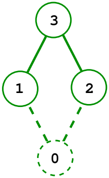

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A more extreme example. Calculate $\mu(\emptyset, 33333)$ when $P = \Lambda$.

The interval $[\emptyset, 33333]$ in P^* has 1906 edges!

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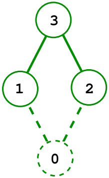

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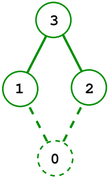

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16

(1)(1+1)(1+1)(1+1)(1+1)

A word or two about the proof

Forman (1995): discrete Morse theory.

Babson & Hersh (2005): discrete Morse theory for order complexes.

If EL-labelings etc. don't work, DMT is worth a try.
(*Take-home message?*)

It boils down to determining which maximal chains are "critical."
Each critical chain contributes $+1$ or -1 to the reduced Euler characteristic / Möbius function.

Not an easy proof: 14 pages with examples.
One subtlety: DMT doesn't give us everything; also utilize classical Möbius function techniques.

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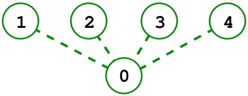
Application 1. Möbius function of subword order (Björner).



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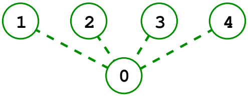

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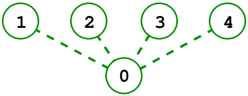

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Application to subword order

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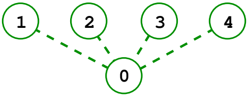

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"Normal embedding" (Björner): whenever $w(j-1) = w(j)$, need jth entry of embedding η to be nonzero.

$$\mu(u,w) = (-1)^{|w|-|u|}(\#\text{ normal embeddings}).$$

More applications

Application 2. Rederive Sagan & Vatter result for μ when P is a rooted forest.

Application 3. In particular, rederive Sagan & Vatter result for μ of composition order.

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Application 4. Rederive Tomie's result for $\mu(1^i, 3^j)$ when $P = \Lambda$.

$$\mu(1^i, 3^j) = [x^{j-i}]T_{i+j}(x) \quad \text{for } 0 \le i \le j$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind.

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

or

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} (2x)^{n-2k}.$$

More applications

Application 5. Tomie's results for augmented Λ.

Application 6. Suppose $\text{rk}(P) \leq 1$. Then any interval $[u, w]$ in P^* is

- shellable;
- homotopic to a wedge of $|\mu(u, w)|$ spheres, all of dimension $\text{rk}(w) - \text{rk}(u) - 2$.

Open problem. What if $\text{rk}(P) \geq 2$?

Summary

- Generalized subword order interpolates between subword order and an order on compositions.
- Simple formula for the Möbius function of P^* for any P. Implies all previously proved cases.
- Proof primarily uses discrete Morse theory.