The Topology of the Permutation Pattern Poset

Peter McNamara and Einar Steingrímsson

Bucknell University — peter.mcnamara@bucknell.edu

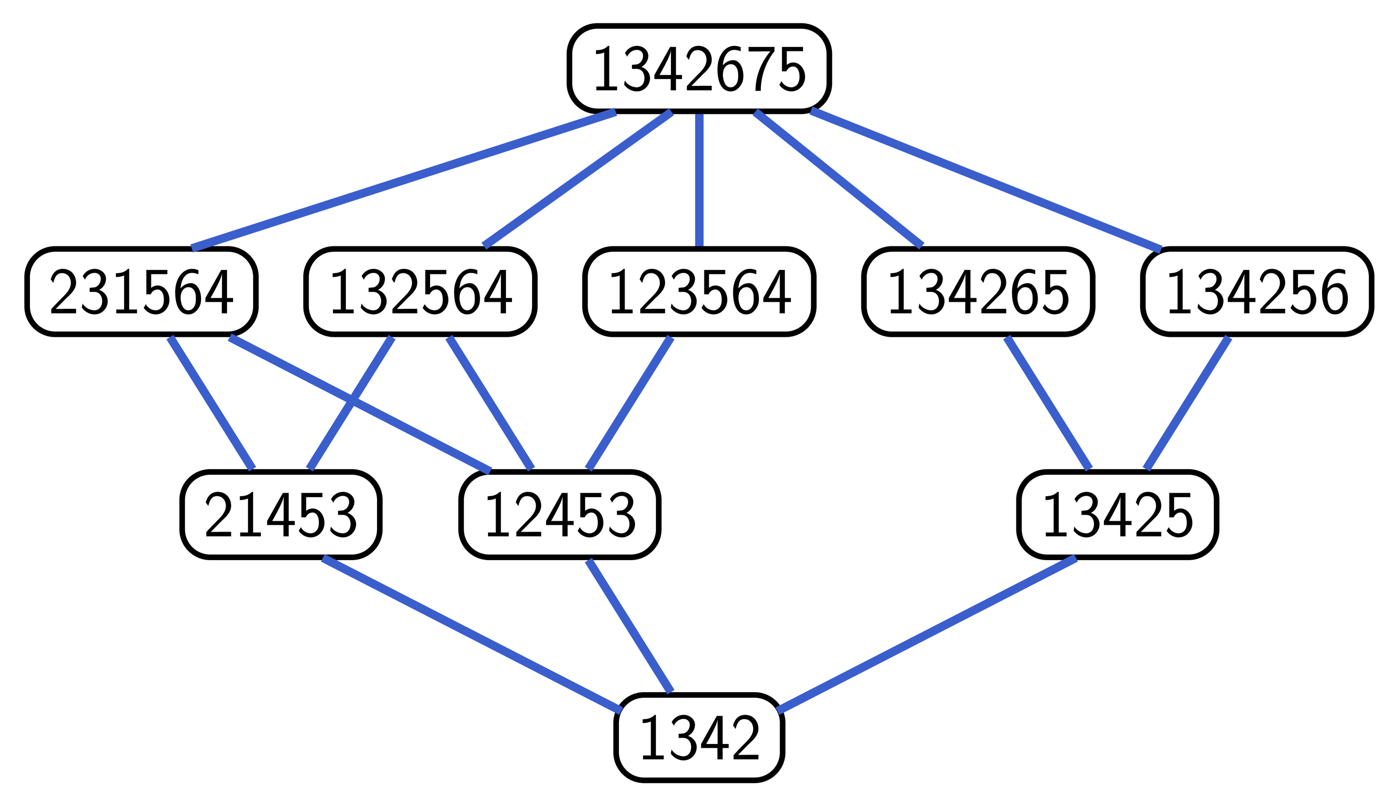
University of Strathclyde — einar@alum.mit.edu

Background & Motivation

Pattern order: order permutations by pattern containment

e.g., **4132** $\leq$ **516423**

Example of a PPP interval



Motivating question [Wilf]

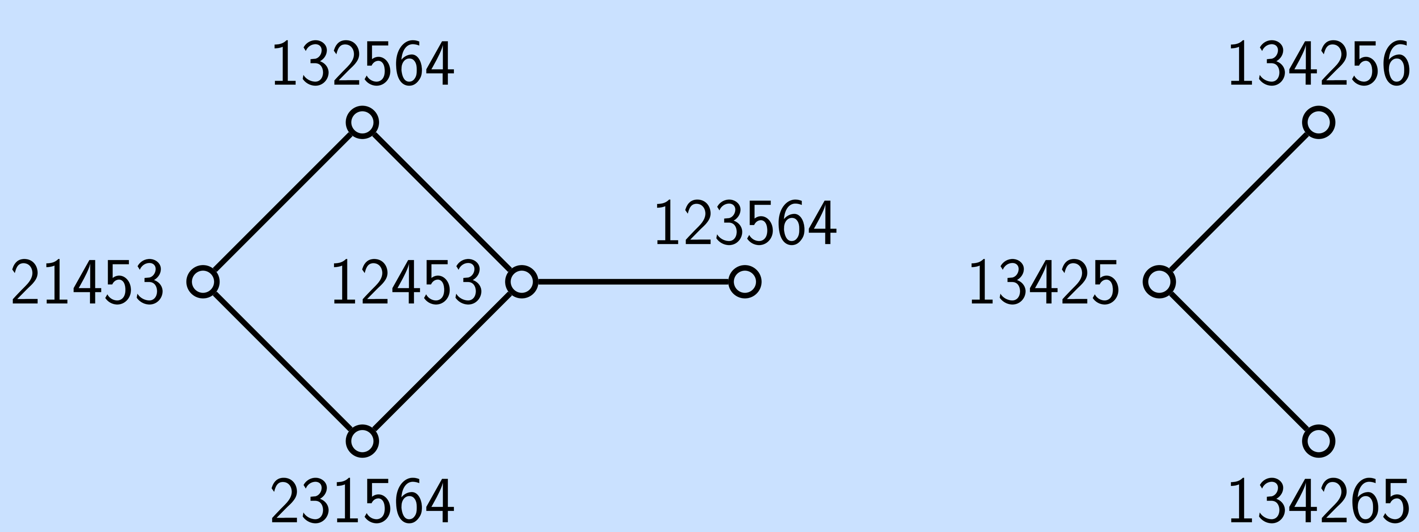
What's the Möbius function of such intervals?

- Sagan & Vatter (2006)
- Steingrímsson & Tenner (2010)
- Burstein, Jelínek, Jelínková & Steingrímsson (2011)
- Smith (2013, 2014)

General case still open.

OUR FOCUS: other topological questions

- Are the open intervals connected?
- Shellable?
- What is their homotopy type?



Main Results

Fact [Björner]:
Non-trivial disconnected subinterval $\Longrightarrow$ not shellable

Theorem (almost all intervals are not shellable)

Fix σ. Randomly choose τ of length n.

$$\lim_{n\to\infty} (\text{Probability that } \Delta(\sigma,\tau) \text{ is shellable}) = 0.$$

In contrast, there's a large class of intervals that **are** shellable.

Definitions.

Direct sum: $21 \oplus 3214 = 215436$.

π is **layered** if it takes the form
$\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k$ with each π_i decreasing.

e.g. $21 \oplus 321 \oplus 321 \oplus 1 = 215438769$

Theorem (all possible layered intervals are shellable)

Suppose σ, τ layered such that $[\sigma, \tau]$ does not contain a non-trivial disconnected subinterval. Then $[\sigma, \tau]$ is shellable.

Proof: CL-shellability (intricately)

More Results

Theorem (augmentation preserves disconnectivity)

Suppose (σ, τ) is disconnected. Then so are $(\alpha \oplus \sigma, \alpha \oplus \tau)$ and $(\sigma \oplus \alpha, \tau \oplus \alpha)$.

Under fairly mild conditions, the new intervals are isomorphic to the original, e.g.

$$(321, 321 \oplus 321) \cong (312 \oplus 321, 312 \oplus 321 \oplus 321)$$

Theorem (a unified Möbius function formula)

Let $\tau = \tau_1 \oplus \tau_2 \oplus \cdots \oplus \tau_t$ be finest. Then

$$\mu(\sigma,\tau) = \sum_{\sigma=\sigma_1\oplus\cdots\oplus\sigma_t} \prod_{1\leq m\leq t} \mu(\sigma_m, \tau_m) + \square$$

where $\square = 1$ if $\sigma_m = \emptyset$ and $\tau_{m-1} = \tau_m$,
and $\square = 0$ otherwise.

Open Problems

- Understand non-shellable intervals without disconnected subintervals.
e.g. $[123, 3416725]$.

- Find a good way to test shellability by computer.

- **Separable permutations:** can be built from 1 by a sequence of direct sums or skew sums.

$$1 \oplus 1 = 12$$
$$(1 \oplus 1) \ominus (1 \oplus 1) = 12 \ominus 12 = 3412$$
$$1 \oplus 3412 = 14523 \quad \text{etc.}$$

Conjecture. Suppose σ, τ separable such that $[\sigma, \tau]$ does not contain a non-trivial disconnected subinterval. Then $[\sigma, \tau]$ is shellable.

- Conjecture. $[\sigma, \tau]$ is always rank unimodal.

Take-home idea

The PPP is hard! But there are substantial classes where results have been pried out in recent years, and almost certainly more is to come.

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