The Möbius function of generalized subword order

Peter McNamara
Bucknell University
(2012/2013 at Trinity College Dublin)

Joint work with:
Bruce Sagan
Michigan State University

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Slides and full paper (*Adv. Math.*) available from
www.facstaff.bucknell.edu/pm040/

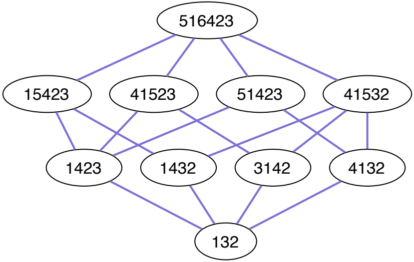
Outline

- Generalized subword order and related posets
- Main result
- Applications

Motivation: Wilf's question

Pattern order: order permutations by pattern containment.

e.g., $132 \leq 5\textcolor{red}{16}42\textcolor{red}{3}$



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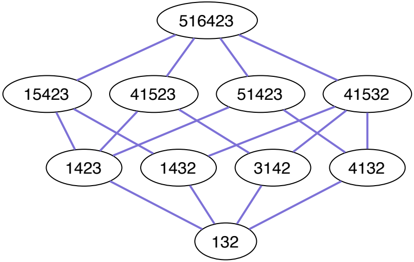

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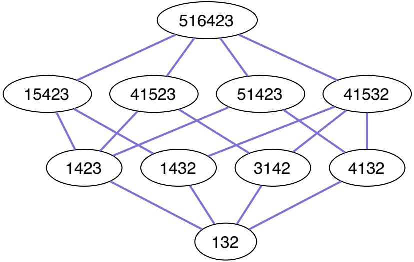

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- Sagan & Vatter (2006)
- Steingrímsson & Tenner (2010)
- Burstein, Jelínek, Jelínková & Steingrímsson (2011)

Still open.

Motivation for generalized subword order

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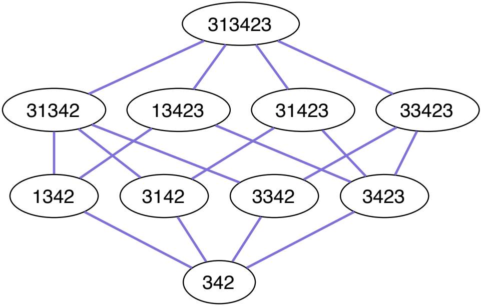
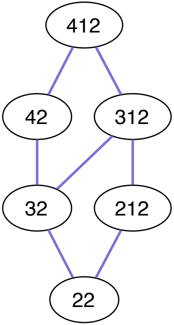
2 partial orders.

1. Subword order.
   A^*: set of finite words over alphabet A.
   $u \leq w$ if u is a subword of w, e.g., 342 $\leq$ 313423.



Motivation for generalized subword order

2. An order on compositions.
$(a_1, a_2, \ldots, a_r) \leq (b_1, b_2, \ldots, b_s)$ if there exists a subsequence $(b_{i_1}, b_{i_2}, \ldots, b_{i_r})$ such that $a_j \leq b_{i_j}$ for $1 \leq j \leq r$.
e.g. $22 \leq 412$.



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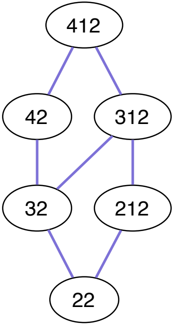

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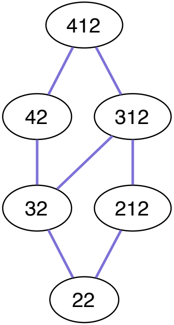

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$2\ 2 \leftrightarrow 21\ 43$

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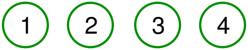
Example 1. If P is an antichain, $u(j) \leq_P w(i_j)$ iff $u(j) = w(i_j)$.



Gives subword order on the alphabet P, e.g., $342 \leq 313423$.

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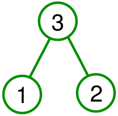
Example 2. If P is the chain below, $u(j) \le_P w(i_j)$ iff $u(j) \le w(i_j)$ as integers.

Gives composition order, e.g. $22 \le 412$.

Key example

Example 3. $P = \Lambda$



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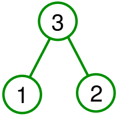

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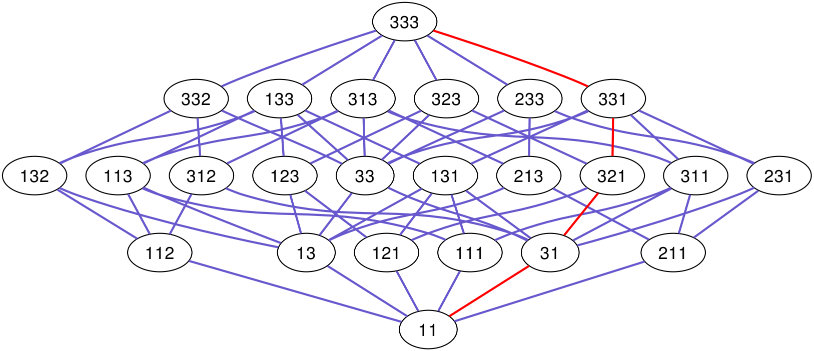

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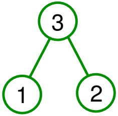
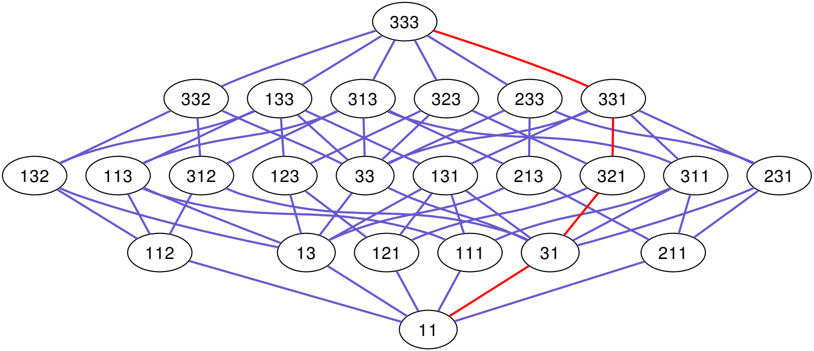


e.g. $\color{red}{1}\color{blue}{1} \le \color{blue}{33}\color{red}{3}$ but $11 \not\le 222$. The interval $[11, 333]$ in P^*:



Definition from Sagan & Vatter (2006); appeared earlier in context of well quasi-orderings [Kruskal, 1972 survey].

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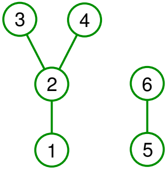

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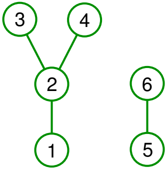

  Includes antichains and chains.
- Sagan & Vatter (2006): when $P = \Lambda$,
  conjecture that $\mu(1^i, 3^j)$ equals certain coefficients of Chebyshev polynomials of the first kind.

  Tomie (2010): proof using methods not easily extendable.

  Our first goal: a more systematic proof.

Main result

P_0: P with a bottom element 0 adjoined.
μ_0: Möbius function of P_0.

Theorem. Let P be a poset so that P_0 is locally finite. Let u and w be elements of P^* with $u \leq w$. Then

$$\mu(u, w) = \sum_{\eta} \prod_{1 \leq j \leq |w|} \begin{cases} \mu_0(\eta(j), w(j)) + 1 & \text{if } \eta(j) = 0 \text{ and} \\ & w(j-1) = w(j), \\ \mu_0(\eta(j), w(j)) & \text{otherwise,} \end{cases}$$

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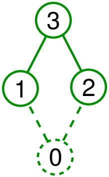

w	$=$	333	333	333
η	$=$	110	101	011
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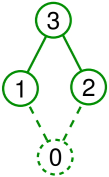

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A more extreme example. Calculate $\mu(\emptyset, 33333)$ when $P = \Lambda$.

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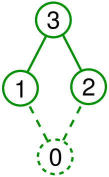

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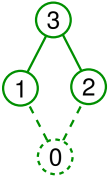

33333
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16

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A word or two about the proof

Forman (1995): discrete Morse theory.

Babson & Hersh (2005): discrete Morse theory for order complexes.

Determine which maximal chains are "critical."
Each critical chain contributes $+1$ or -1 to the reduced Euler characteristic / Möbius function.

Take-home message? If the usual methods for determining Möbius functions don't work, try DMT.

Not an easy proof: 14 pages with examples.
One subtlety: DMT doesn't give us everything; also utilize classical Möbius function techniques.

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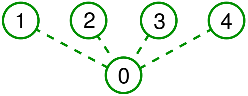

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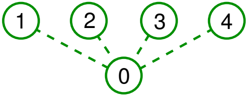

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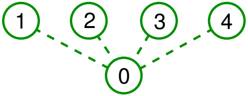

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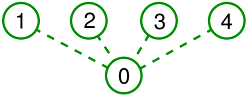

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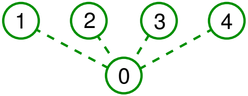

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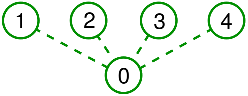

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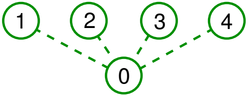

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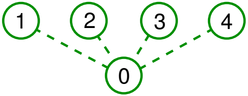

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$$\mu(u,w) = (-1)^{|w|-|u|}(\#\text{ normal embeddings}).$$

More applications

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Application 4. Rederive Tomie's result for $\mu(1^i, 3^j)$ when $P = \Lambda$.

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where $T_n(x)$ is the Chebyshev polynomial of the first kind.

More applications

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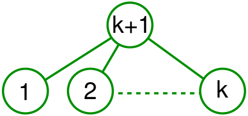
Application 5. Tomie's results for augmented Λ.



A topological application

Application 6. Suppose $\text{rk}(P) \leq 1$. Then any interval $[u, w]$ in P^* is

- shellable;
- homotopic to a wedge of $|\mu(u, w)|$ spheres, all of dimension $\text{rk}(w) - \text{rk}(u) - 2$.

Open problem. What if $\text{rk}(P) \geq 2$?

Summary

- Generalized subword order interpolates between subword order and an order on compositions.
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- Proof primarily uses discrete Morse theory.
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Thanks!

ありがとう

$[\emptyset, 33333]$ when $P = \lambda$