Background and Motivation

Sequence: $(a_k) = a_0, a_1, ..., a_n$. Let $a_i = 0$ for i < 0 and i > n.

 (a_k) is log-concave if $a_k^2 \ge a_{k-1}a_{k+1}$ for all k. e.g., rows of Pascal's triangle

Infinite log-concavity

L-operator on sequences defined by $\mathcal{L}(a_k) = (b_k)$ where $b_k = a_k^2 - a_{k-1}a_{k+1}$. e.g.,

- $(a_k) = 0, 1, 4, 6, 4, 1, 0$ $\mathcal{L}(a_k) = 0, 1, 10, 20, 10, 1, 0$ $\mathcal{L}^{2}(a_{k}) = 0, 1, 80, 300, 80, 1, 0$
- (a_k) is *i*-fold log-concave if $\mathcal{L}^i(a_k)$ is a nonnegative sequence. (So log-concavity = 1-fold log-concavity.)
- (a_k) is infinitely log-concave if $\mathcal{L}^i(a_k)$ is nonnegative for all i > 0.

Motivating conjecture [Boros and Moll]

The rows of Pascal's triangle are infinitely log-concave.

Kauers and Paule's result

The rows of Pascal's triangle are 5-fold log-concave.

Infinite Log-Concavity: Developments and Conjectures Peter McNamara and Bruce Sagan Michigan State University Bucknell University peter.mcnamara@bucknell.edu sagan@math.msu.edu

Main Result

Main idea

Log-concavity is not preserved under \mathcal{L} . e.g., $\mathcal{L}(4, 5, 4) = 16, 9, 16.$

So let $r \ge 1$. Say that (a_k) is *r*-factor log-concave if

 $a_{k}^{2} \geq r a_{k-1}a_{k+1}$.

Note that *r*-factor log-concavity implies log-concavity.

Theorem (*r*-factor log-concavity persists)

Let (a_k) be a nonnegative sequence and let $r=\frac{3+\sqrt{5}}{2}\approx 2.618$

If (a_k) is *r*-factor log-concave, so is $\mathcal{L}(a_k)$. Thus (a_k) is infinitely log-concave.

Corollary

The row $\binom{n}{k}_{k>0}$ of Pascal's triangle is infinitely log-concave for all n < 1450.

Proof: Using a computer, repeatedly apply \mathcal{L} to the row until it becomes *r*-factor log-concave.

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1
1	6	15	20	15	6
1	7	21	35	35	21
1	8	28	56	70	56

28 8

Conjecture (columns)

The columns of Pascal's triangle are infinitely log-concave.

Conjecture (diagonals)

The diagonal $\binom{n+mu}{mv}_{m\geq 0}$ is infinitely log-concave if u < v.

The **q**-analogue $\left(\begin{bmatrix} n \\ k \end{bmatrix} \right)_{k>0}$ of a row is not even 2-fold log-concave. (Now "nonnegative" means all coefficients are nonnegative.)

Conjecture (q-analogue of a column)

 $\binom{n}{k}_{n>k}$ is infinitely log-concave for all fixed **k**.

Theorem (symmetric functions)

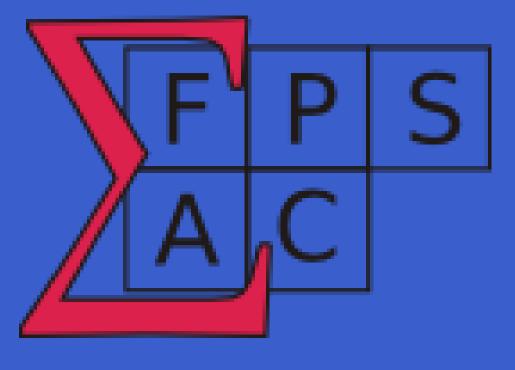
 $(h_k)_{k\geq 0}$ is 3-fold log-concave but is not 4-fold log-concave.

For $(a_k) = a_0, a_1, ..., a_n$ with $a_i \ge 0$ let $p[a_k] = a_0 + a_1 x + \cdots + a_n x^n$.

If $p[a_k]$ has all real roots, then so does $p[\mathcal{L}(a_k)].$

Take-home idea

Infinite log-concavity is a natural concept deserving further study.



Other Directions

Conjecture [Fisk, M-S, Stanley] (real roots)