Positivity Questions for Cylindric Skew Schur Functions

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Slides and full paper available from
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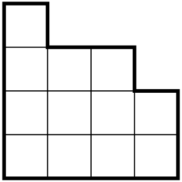
Outline

- What are cylindric skew Schur functions?
- When are they Schur-positive?
- An expansion in terms of skew Schur functions
- A Schur-positivity conjecture

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$

- Young diagram.
  Example:
  $\lambda = (4, 4, 3, 1)$



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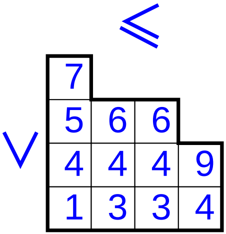
- Semistandard Young tableau (SSYT)



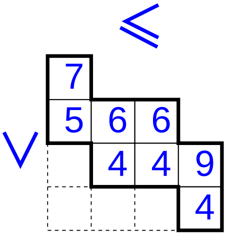
The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

Example

$$s_{4431} = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \cdots .$$

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- μ fits inside λ.
- Young diagram.
  Example:
  $\lambda/\mu = (4,4,3,1)/(3,1)$
- Semistandard Young tableau (SSYT)



The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

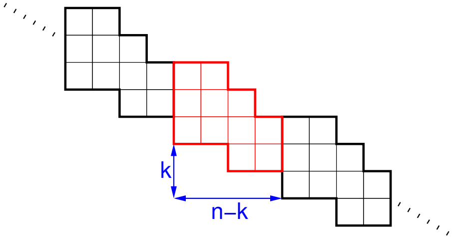
$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \cdots .$$

Example

$$s_{4431/31} = \quad x_4^3 x_5 x_6^2 x_7 x_9 + \cdots .$$

Cylindric skew Schur functions

- Infinite skew shape C
- Invariant under translation
- Identify (a, b) and $(a + n - k, b - k)$.



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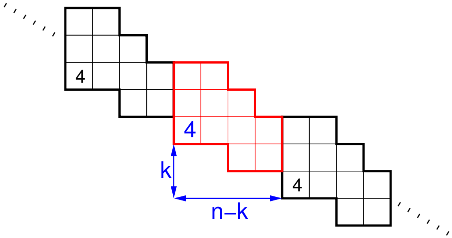

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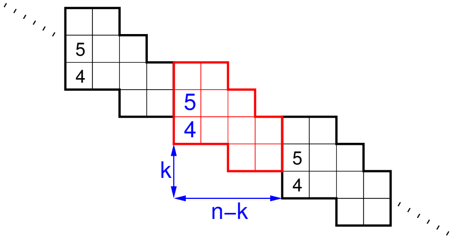

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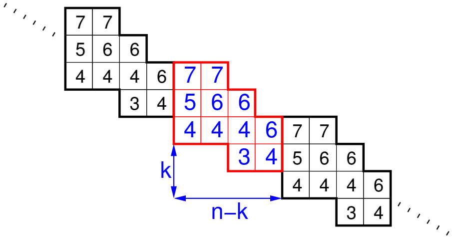

- Entries weakly increase in each row
  Strictly increase up each column
- Alternatively: SSYT with relations between entries in first and last columns
- Cylindric skew Schur function:

$$s_C(x) = \sum_T x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

$$\text{e.g. } s_C(x) = x_3 x_4^4 x_5 x_6^3 x_7^2 + \cdots .$$

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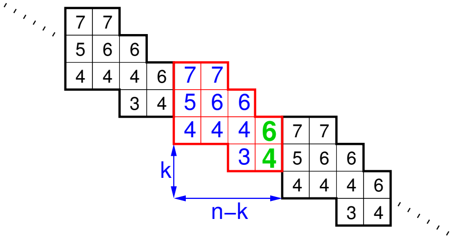

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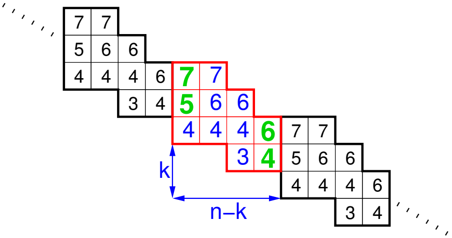

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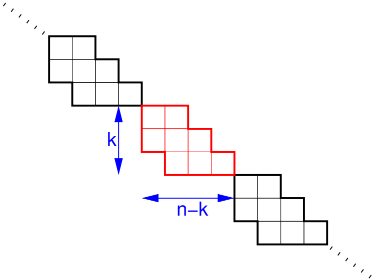
Skew shapes are cylindric skew shapes...

... and so skew Schur functions are cylindric skew Schur functions.

Example



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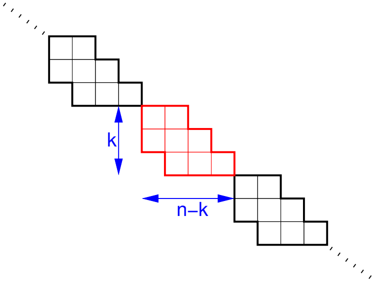

- Gessel, Krattenthaler: *"Cylindric partitions,"* 1997.
- Bertram, Ciocan-Fontanine, Fulton: *"Quantum multiplication of Schur polynomials,"* 1999.
- Postnikov: *"Affine approach to quantum Schubert calculus,"* `math.CO/0205165`.
- Stanley: *"Recent developments in algebraic combinatorics,"* `math.CO/0211114`.

Motivation: Positivity of Gromov-Witten invariants

In $H^*(\mathrm{Gr}_{kn})$,

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq k \times (n-k)} c_{\mu\nu}^{\lambda} \sigma_\lambda.$$

In $QH^*(\mathrm{Gr}_{kn})$,

$$\sigma_\mu * \sigma_\nu = \sum_{d \geq 0} \sum_{\lambda \subseteq k \times (n-k)} q^d C_{\mu\nu}^{\lambda,d} \sigma_\lambda.$$

$C_{\mu\nu}^{\lambda,d}$ = 3-point Gromov-Witten invariants
= $\#\{$rational curves of degree d in Gr_{kn} that meet $\tilde{\Omega}_\mu$, $\tilde{\Omega}_\nu$ and $\tilde{\Omega}_{\lambda^\vee}\}$.

Example

$$C_{\mu,\nu}^{\lambda,0} = c_{\mu\nu}^{\lambda}.$$

Key point: $C_{\mu\nu}^{\lambda,d} \geq 0$.

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Key point: $C_{\mu\nu}^{\lambda,d} \geq 0$.
"Fundamental open problem": Find an algebraic or combinatorial proof of this fact.

Connection with cylindric skew Schur functions

Theorem (Postnikov)

$$s_{\mu/d/\nu}(x_1,\dots,x_k) = \sum_{\lambda \subseteq k\times(n-k)} C_{\mu\nu}^{\lambda,d} s_\lambda(x_1,\dots,x_k).$$

Conclusion: Want to understand the expansions of cylindric skew Schur functions into Schur functions.

Theorem (Postnikov)

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Corollary

$s_{\mu/d/\nu}(x_1, \ldots, x_k)$ *is Schur-positive.*

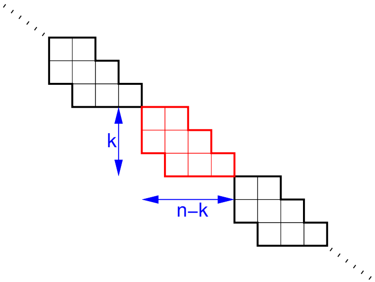
Known: $s_{\mu/d/\nu}(x_1, x_2, \ldots) \equiv s_{\mu/d/\nu}(x)$ need not be Schur-positive.

Example

If $s_{\mu/d/\nu} = s_{22} + s_{211} - s_{1111}$, then $s_{\mu/d/\nu}(x_1, x_2, x_3)$ is Schur-positive.

(In general: $s_\lambda(x_1, \ldots, x_k) \neq 0 \Leftrightarrow \lambda$ has at most k parts.)

When is a cylindric skew Schur function Schur-positive?



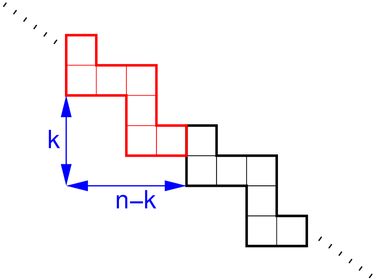
Theorem (McN.)

For any cylindric skew shape C,

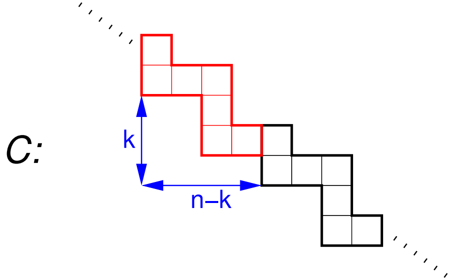
$$s_C(x_1, x_2, \ldots) \textit{ is Schur-positive} \Leftrightarrow C \textit{ is a skew shape}.$$

Equivalently, if C is a non-trivial cylindric skew shape, then $s_C(x_1, x_2, \ldots)$ is not Schur-positive.

C:



$$s_C(x_1, x_2, \ldots) = \sum_{\lambda \subseteq k \times (n-k)} c_\lambda s_\lambda + s_{(n-k,1^k)} - s_{(n-k-1,1^{k+1})} + s_{(n-k-2,1^{k+2})} - \cdots + (-1)^{n-k} s_{(1^n)}.$$



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Schur-positive in $k+1$ variables.

Not Schur-positive in $\geq k+2$ variables.

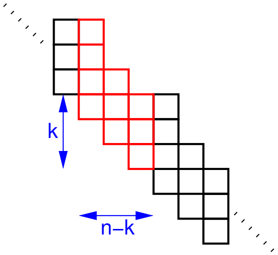
General cylindric skew shape: $\geq k+2+\ell$ variables.

Shapes in Postnikov's theorem: $\geq 2k+1$ variables.

Formula: cylindric skew Schur functions as signed sums of skew Schur functions

Idea for formulation: Bertram, Ciocan-Fontanine, Fulton
Uses result of Gessel, Krattenthaler

Example



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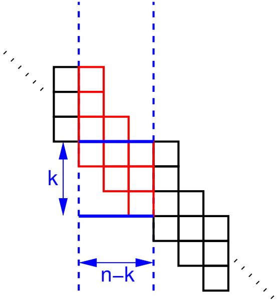

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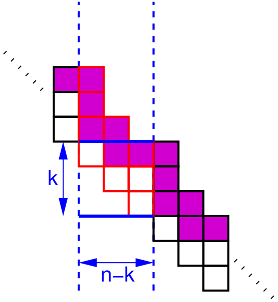

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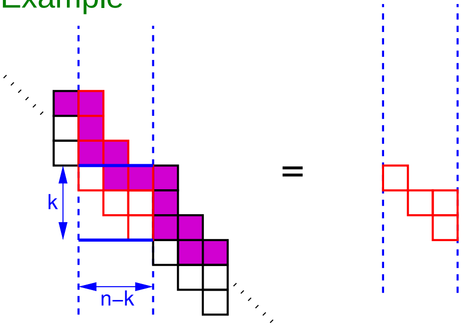

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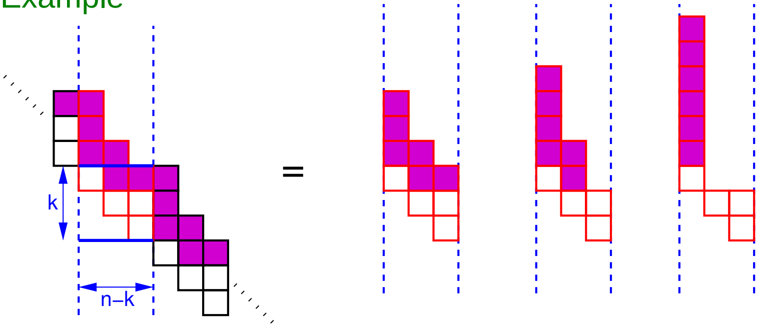

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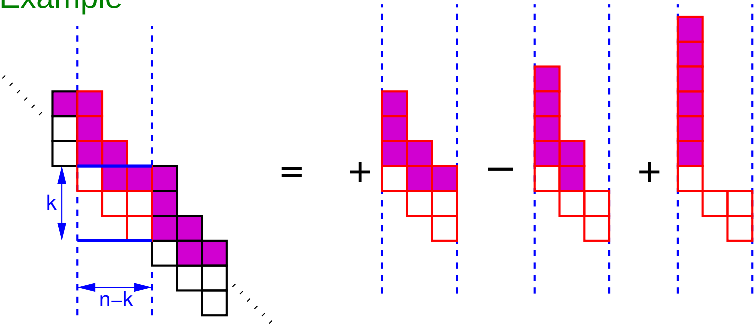

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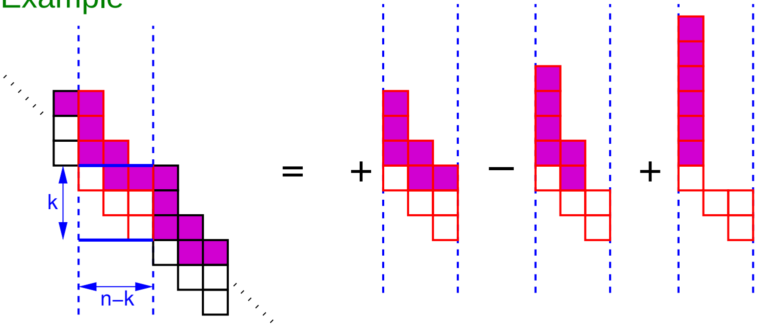
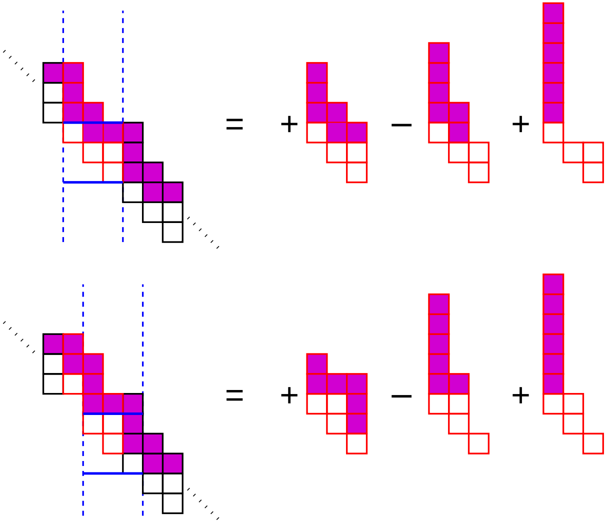


$$s_C = s_{333211/21} - s_{3322111/21} + s_{331111111/21}$$

$$= s_{3331} + s_{3322} + s_{33211} + s_{322111} + s_{31111111}$$

$$- s_{222211} - s_{2221111} + s_{22111111} + s_{211111111}.$$

Quick consequence: lots of skew Schur function identities



Shouldn't cylindric skew Schur functions be Schur-positive *in some sense*?

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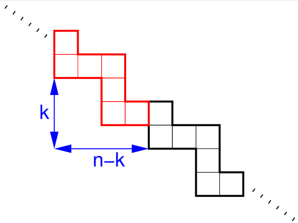

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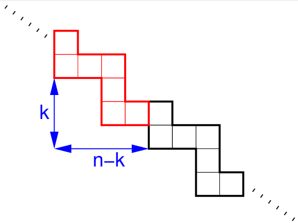
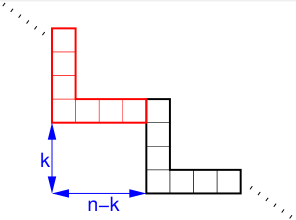


H:



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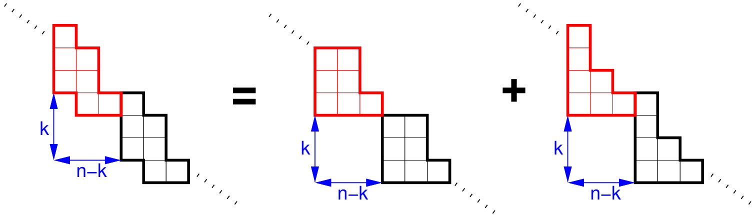
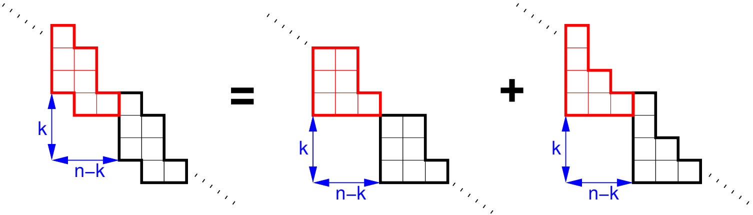
In fact,

$$s_C(x_1, x_2, \ldots) = \sum_{\lambda \subseteq k \times (n-k)} c_\lambda s_\lambda + s_H.$$

s_C: cylindric *skew* Schur function
s_H: cylindric Schur function
We say that s_C is cylindric Schur-positive.

A Conjecture





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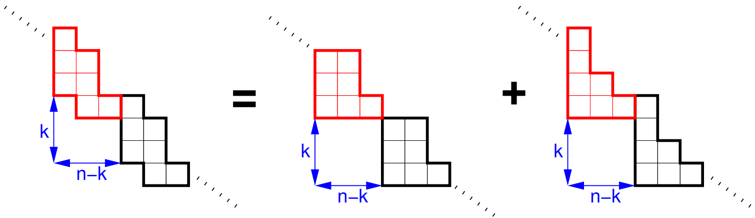

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Theorem (McN.)

The cylindric Schur functions corresponding to a given translation $(-n+k, +k)$ are linearly independent.

Theorem (McN.)

If s_C can be written as a linear combination of cylindric Schur functions with the same translation as C, then s_C is cylindric Schur-positive.

Summary of results

- Classification of those cylindric skew Schur functions that are Schur-positive.
- Full knowledge of negative terms in Schur expansion of ribbons.
- Expansion of any cylindric skew Schur function into skew Schur functions.
- Conjecture and evidence that every cylindric skew Schur function is cylindric Schur-positive.