The Schur-Positivity Poset

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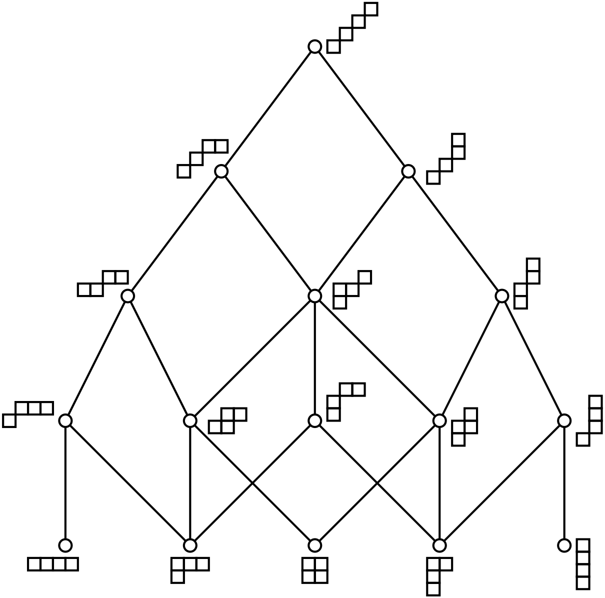
Slides and paper available from
`www.facstaff.bucknell.edu/pm040/`

Outline

- Introduction to the Schur-positivity poset
- Some relevant results
- Restriction to multiplicity-free ribbons
- Lattice structure in more detail

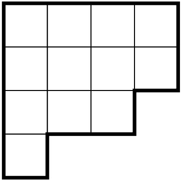
Preview

$n = 4$



Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- Young diagram.
  Example:
  $\lambda \quad = (4, 4, 3, 1)$



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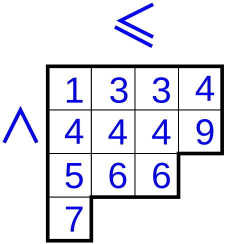
- Semistandard Young tableau (SSYT)



The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

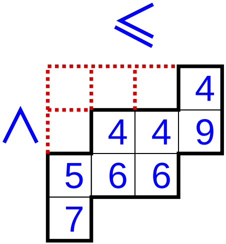
$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

Example

$s_{4431} = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \cdots .$

Skew Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- μ fits inside λ.
- Young diagram.
  Example:
  $\lambda/\mu = (4, 4, 3, 1)/(3, 1)$
- Semistandard Young tableau (SSYT)



The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

Example

$$s_{4431/31} = \qquad x_4^3 x_5 x_6^2 x_7 x_9 + \cdots .$$

Some reasons why we care

- Skew Schur functions are symmetric in the variables $x = (x_1, x_2, \ldots)$.
- The Schur functions form a basis for the algebra of symmetric functions (over $\mathbb{Q}$, say).
- Connections with Algebraic Geometry, Representation Theory.

Schur-positivity

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}.$$

Littlewood-Richardson rule:
$c^{\lambda}_{\mu\nu}$ is the number of SSYT of shape λ/μ and content ν whose *reverse reading word* is a *ballot sequence*.

Key: $c^{\lambda}_{\mu\nu} \geq 0$.

$s_{\lambda/\mu}$ is Schur-positive (i.e. coefficients in Schur expansion are all non-negative).

Connection between Schur-positivity and representation theory of S_n.

Big Question

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}.$$

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Definition

Let A, B be skew shapes. We say that

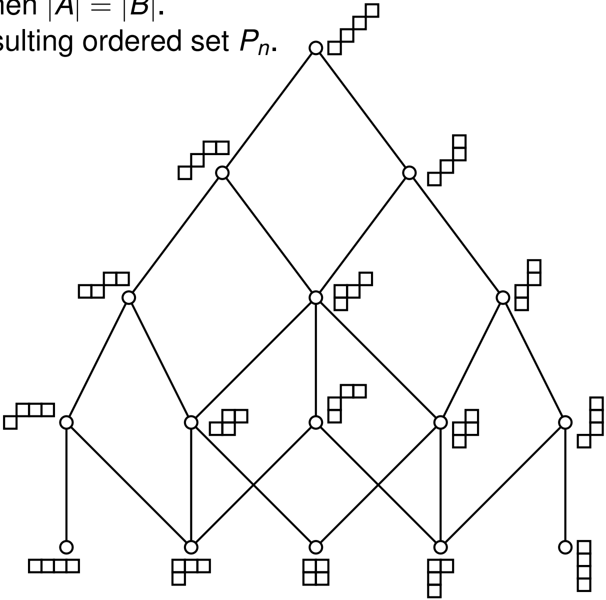
$$A \leq_s B \quad \text{if} \quad s_B - s_A$$

is Schur-positive.

Goal: Characterize the Schur-positivity order $\leq_s$ in terms of skew shapes.

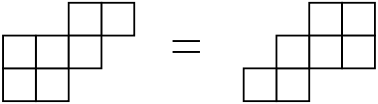
Example

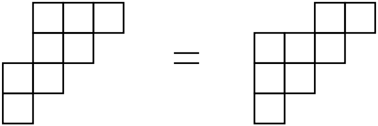
If $A \leq_s B$ then $|A| = |B|$.
Call the resulting ordered set P_n.
Then P_4:

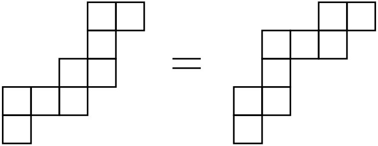


First things first

$\leq_s$ is not yet anti-symmetric. So identify skew shapes such as

1. 

2. 

3. 

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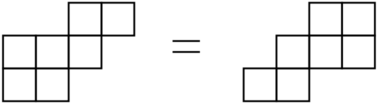
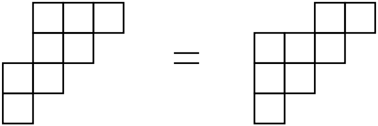
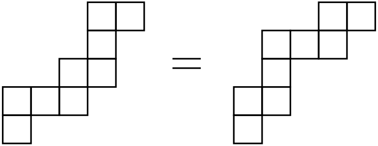
1. 

2. 

3. 

Definition

A ribbon is a connected skew shape containing no 2×2 rectangle.

Indexed as 21231 and 23121.

Relevant results: skew Schur equality

Question: When is $s_A = s_B$?

- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004):

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  Complete classification of equality of ribbon Schur functions

- Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006)
- McN., Steph van Willigenburg (2006)

Enough for our purposes: we can consider P_n to be a poset.

Relevant results: Necessary *or* sufficient conditions

Sufficient conditions for $A \leq_s B$:

- A. Lascoux, B. Leclerc, J.-Y. Thibon (1997)
- A. Okounkov (1997)
- S. Fomin, W. Fulton, C.-K. Li, Y.T. Poon (2003)
- A.N. Kirillov (2004)
- T. Lam, A. Postnikov, P. Pylyavskyy (2005)
- F. Bergeron, R. Biagioli, M. Rosas (2006)
- ...

Necessary conditions for $A \leq_s B$:

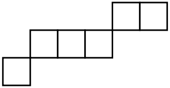
- McN. (2007)

Relevant results: special classes of skew shapes

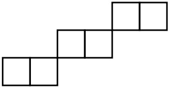
- Macdonald's "Symmetric functions and Hall polynomials": For horizontal strips, $A \leq_s B$ if and only if

$$\text{row lengths of } A \geq \text{row lengths of } B$$

in dominance order.



$\leq_s$



P_n restricted to horizontal strips: (dual of the) dominance lattice.

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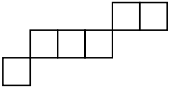
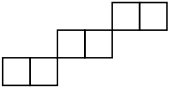
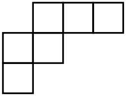
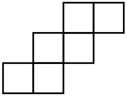
   $\leq_s$ 

  P_n restricted to horizontal strips: (dual of the) dominance lattice.

- Ron King, Trevor Welsh, Steph van Willigenburg (2007): For ribbons with decreasing row lengths and equal numbers of rows, same is true.

   $\leq_s$ 

What about general ribbons?

Suffices to fix $\#$boxes and $\#$rows.

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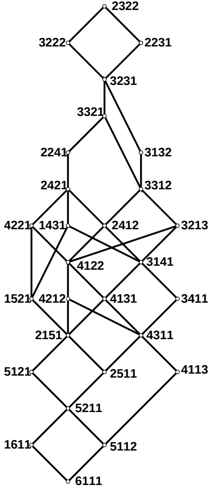
Bad news: *Ribbons*$(9, 4)$:



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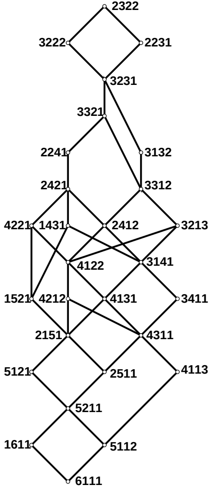

Crux of the matter: What's a feasible yet interesting subposet?

Multiplicity-free ribbons

Definition

A skew shape A is said to be multiplicity-free if, when s_A is expanded as a linear combination of Schur functions, each coefficient is 0 or 1.

Question: When is a ribbon A multiplicity-free?

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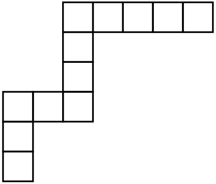
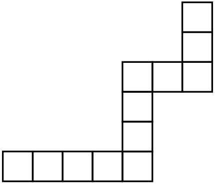
Question: When is a ribbon A multiplicity-free?

Answer: (Thomas-Yong 2005, Gutschwager 2006) When A has at most two rows with more than one box, and at most two columns with more than one box.

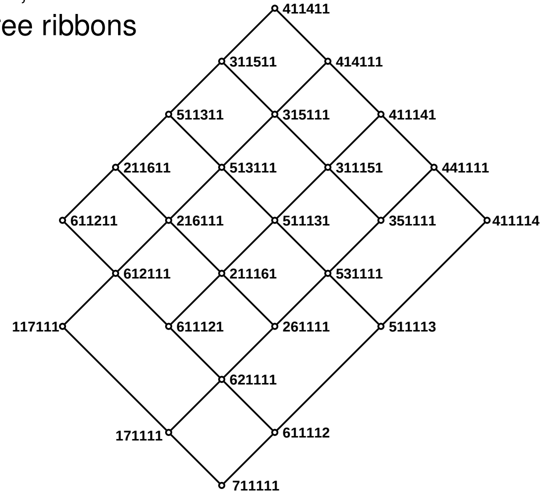




Multiplicity-free Schur expansions correspond to multiplicity-free representations; survey article by Roger Howe.

Subposet of multiplicity-free ribbons

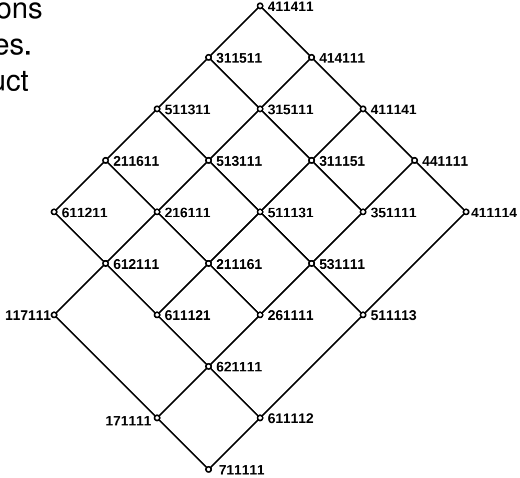
Theorem (roughly stated) (McN., van Willigenburg, 2007) For a given number of boxes and rows,
the poset of multiplicity-free ribbons
is always of the form



and is a convex subposet of the appropriate Schur-positivity poset.

A word or two about the proof

- Proof uses standard Littlewood-Richardson rule to prove lots of equalities.
- Actually know difference along each edge.
- Uses necessary conditions to prove incomparabilities.
- Why not exactly a product of two chains?



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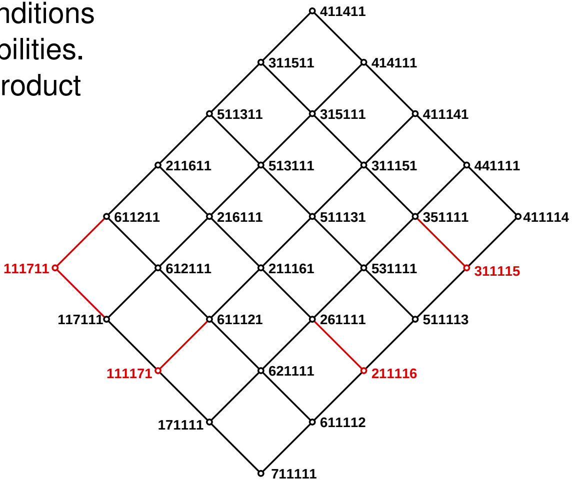
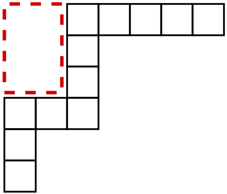


Why a product of two chains?

Fix #boxes=n, #rows=r.
Multiplicity-free ribbons are indexed by rectangles.
Example $n = 12$, $r = 6$. The multiplicity-free ribbon $[3, 2]$:



Define two total orders on the positive integers:

- Row order: $r-1 \;<_r\; 1 \;<_r\; r-2 \;<_r\; 2 \;<_r\; \cdots <_r \lfloor \frac{r}{2} \rfloor$.
  e.g. $5 <_r 1 <_r 4 <_r 2 <_r 3$.
- Column order:
  $N-r \;<_c\; 1 \;<_c\; N-r-1 \;<_c\; 2 <_c \cdots <_c \lfloor \frac{N-r+1}{2} \rfloor$.
  e.g. $6 <_c 1 <_c 5 <_c 2 <_c 4 <_c 3$.

Theorem (McN., van Willigenburg, 2007)

$$[a, b] \leq_s [a', b'] \quad \text{if and only if} \quad a \leq_r a' \text{ and } b \leq_c b'.$$

Example

Row order:
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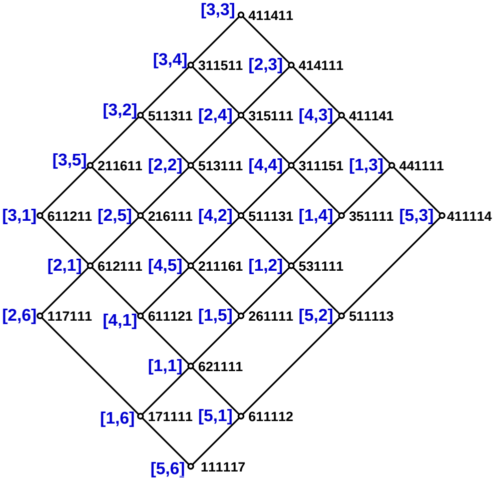
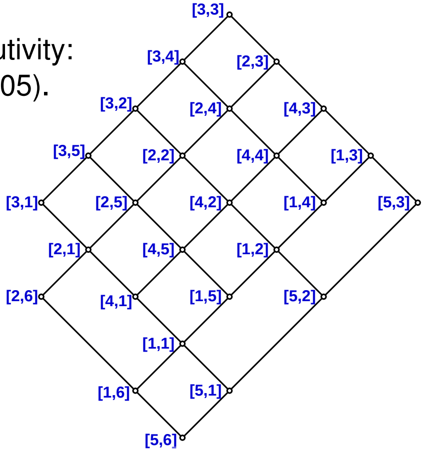
Column order:
$6 <_c 1 <_c 5 <_c 2 <_c 4 <_c 3$.



Lattice-theoretic properties

- Schur-positivity-poset of multiplicity-free ribbons is a lattice.
- Not distributive.
- Ungraded analogue of distributivity: trim lattice (Hugh Thomas, 2005). Stronger than extremal lattice (George Markowsky, 1992).



The last word: Even though the full Schur-positivity poset P_n seems unstructured, the subposet of multiplicity-free ribbons is intelligible and __________.