Inequalities among symmetric polynomials

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Slides and paper available from
`www.facstaff.bucknell.edu/pm040/`

What is algebraic combinatorics anyhow?

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Define combinatorics

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Algebraic combinatorics:

The use of techniques from algebra, topology, and geometry in the solution of combinatorial problems, or the use of combinatorial methods to attack problems in these areas [Billera, Björner, Greene, Simion, Stanley, 1999].

Outline

- Symmetric polynomials/functions
- Skew Schur functions
- Relationships among skew Schur functions
- The quasisymmetric insight

What are symmetric functions?

Definition.

A symmetric polynomial is a polynomial that is invariant under any permutation of its variables $x_1, x_2, \ldots x_n$.

Example.

- $x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^2 x_3 + x_3^2 x_1 + x_3^2 x_2$
  is a symmetric polynomial in x_1, x_2, x_3.

Definition.

A symmetric function is a formal power series that is invariant under any permutation of its (infinite set of) variables $x = (x_1, x_2, \ldots)$.

Examples.

- $\sum_{i \neq j} x_i^2 x_j$ is a symmetric function.
- $\sum_{i < j} x_i^2 x_j$ is not symmetric.

Bases for the symmetric functions

Fact: The symmetric functions form a vector space.
What is a possible basis?

- Monomial symmetric functions: Start with a monomial:

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Given a *partition* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, e.g. $\lambda = (7, 4, 4)$,

$$m_\lambda = \sum_{\substack{i_1, \ldots, i_\ell \\ \text{distinct}}} x_{i_1}^{\lambda_1} \ldots x_{i_\ell}^{\lambda_\ell}.$$

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- Elementary symmetric functions, e_λ.
- Complete homogeneous symmetric functions, h_λ.
- Power sum symmetric functions, p_λ.

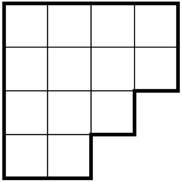
Combinatorial interest: for degree n, dimension = #partitions of n.

Typical questions: Prove they are bases, convert between bases, ...

Schur functions

Cauchy, 1815.

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.
- Young diagram.
  Example: $\lambda = (4, 4, 3, 2)$.



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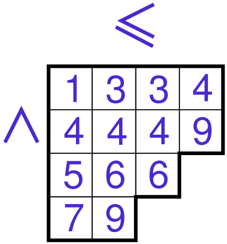

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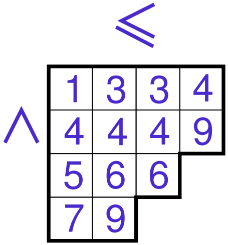

The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_\lambda = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

Example.
$s_{4432} = x_1^1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9^2 + \cdots .$

Example.

$$\leqslant$$

$$\wedge\ \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Hence

$$\begin{aligned} s_{21}(x_1, x_2, x_3) &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 \\ &\quad + 2x_1 x_2 x_3 \\ &= m_{21}(x_1, x_2, x_3) + 2m_{111}(x_1, x_2, x_3). \end{aligned}$$

Facts:

- Schur functions are symmetric functions.
- They form an orthonormal basis: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.

Question. Why do we really care about Schur functions?

But first...

Schur functions

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- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.
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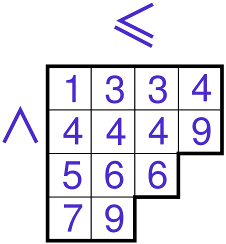

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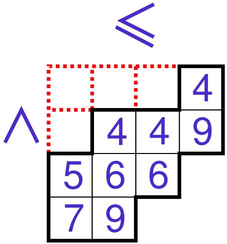
$s_{4432} \quad = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9^2 + \cdots .$

H. Nägelsbach (1871); Craig Aitken (1929)

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.
- μ fits inside λ.
- Young diagram.
  Example: $\lambda/\mu = (4,4,3,2)/(3,1)$
- Semistandard Young tableau (SSYT).



The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

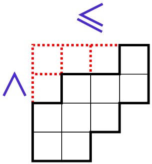
$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

Example.

$$s_{4432/31} = \qquad x_4^3 x_5 x_6^2 x_7 x_9^2 + \cdots .$$

Skew Schur functions

- Skew Schur functions are symmetric functions.



- Conjecture [Stanley, 1972]. Any other shapes give non-symmetric functions.

- There are too many skew Schur functions to form a basis.
- Our interest: What are the relationships among them?
- $$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}$$

$c^{\lambda}_{\mu\nu}$: *Littlewood–Richardson coefficients*

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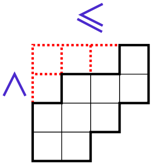

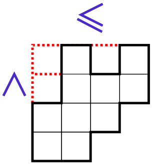

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$c^{\lambda}_{\mu\nu}$: *Littlewood–Richardson coefficients*

s_λ and $c^\lambda_{\mu\nu}$ are superstars!

1. Representation Theory of S_n:

$$(S^\mu \otimes S^\nu) \uparrow^{S_n} = \bigoplus_\lambda c^\lambda_{\mu\nu} S^\lambda, \quad \text{so} \quad \chi^\mu \cdot \chi^\nu = \sum_\lambda c^\lambda_{\mu\nu} \chi^\lambda.$$

   We also have that $s_\lambda =$ the Frobenius characteristic of χ^λ.

2. Representations of $\mathrm{GL}(n, \mathbb{C})$:
   $s_\lambda(x_1, \ldots, x_n) =$ the character of the irreducible rep. V^λ.

$$V^\mu \otimes V^\nu = \bigoplus c^\lambda_{\mu\nu} V^\lambda.$$

3. Algebraic Geometry: Schubert classes σ_λ form a linear basis for $H^*(\mathrm{Gr}_{kn})$. We have

$$\sigma_\mu \sigma_\nu = \sum_{\lambda \subseteq k \times (n-k)} c^\lambda_{\mu\nu} \sigma_\lambda.$$

   Thus $c^\lambda_{\mu\nu}$ = number of points of Gr_{kn} in $\tilde{\Omega}_\mu \cap \tilde{\Omega}_\nu \cap \tilde{\Omega}_{\lambda^\vee}$.

There's more!

4. Linear Algebra: When do there exist Hermitian matrices A, B and $C = A + B$ with eigenvalue sets μ, ν and λ, respectively?

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   When $c^{\lambda}_{\mu\nu} > 0$. (Heckman, Klyachko, Knutson, Tao.)

There's more!

4. Linear Algebra: When do there exist Hermitian matrices A, B and $C = A + B$ with eigenvalue sets μ, ν and λ, respectively?

   When $c_{\mu\nu}^{\lambda} > 0$. (Heckman, Klyachko, Knutson, Tao.)

By 1, 2 or 3 we get:

$$c_{\mu\nu}^{\lambda} \geq 0.$$

Consequence:
We say that $s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}$ is a Schur-positive function.

Want a combinatorial proof that $c_{\mu\nu}^{\lambda} \geq 0$: "They must count something!"
[Littlewood–Richardson rule]

Summary so far

- Symmetric functions: invariant until any permutation of their variables $x_1, x_2, \ldots$.
- Schur functions: (most?) important basis for the symmetric functions.
- Skew Schur functions are Schur-positive.
- (Skew) Schur functions have a beautiful combinatorial definition in terms of tableaux.

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- Symmetric functions: invariant until any permutation of their variables $x_1, x_2, \ldots$.
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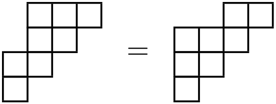
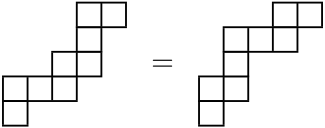
Our focus: What are the relationships among skew Schur functions?

The equality question

s_A: the skew Schur function for the skew shape A.

Wide Open Question. When is $s_A = s_B$?
Determine necessary and sufficient conditions on *shapes* of A and B.





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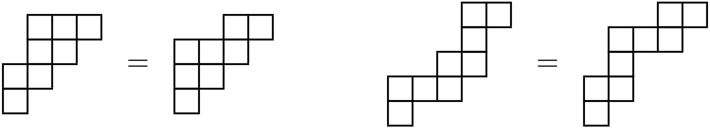

- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004)
- John Stembridge (2004)
- Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006)
- McN., Steph van Willigenburg (2006)
- Christian Gutschwager (2008)

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General idea: the overlaps among rows must match up.

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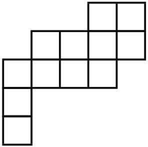
Definition [Reiner, Shaw, van Willigenburg]. For a skew shape A, let $\text{overlap}_k(i)$ be the number of columns occupied in common by rows $i, i+1, \ldots, i+k-1$.
Then $\text{rows}_k(A)$ is the weakly decreasing rearrangement of $(\text{overlap}_k(1), \text{overlap}_k(2), \ldots)$.

Example.

$A =$ 

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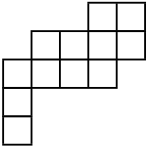
$A =$ 

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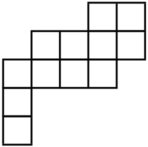
$A =$ 

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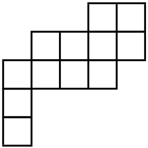
$A =$ 

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- $\text{rows}_3(A) = 11$.

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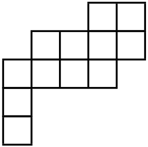
$A =$ 

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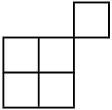
- $\text{rows}_3(A) = 11$.
- $\text{rows}_k(A) = \emptyset$ for $k > 3$.

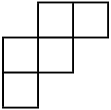
Theorem [RSvW]. Let A and B be skew shapes. If $s_A = s_B$, then

$$\text{rows}_k(A) = \text{rows}_k(B) \text{ for all } k.$$

Example.



$\neq$



$$(221, 2) \neq (221, 11)$$

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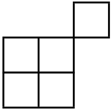
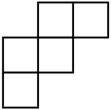
 $\neq$ 

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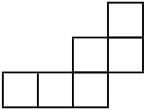
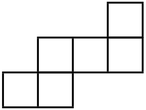
Converse is not true:

 $\neq$ 

$$(321, 11) \qquad (321, 11)$$

Our main interest: inequalities.

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}.$$

When is $s_{\lambda/\mu} - s_{\sigma/\tau}$ Schur-positive?

Schur-positivity order

Our main interest: inequalities.

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}.$$

When is $s_{\lambda/\mu} - s_{\sigma/\tau}$ Schur-positive?

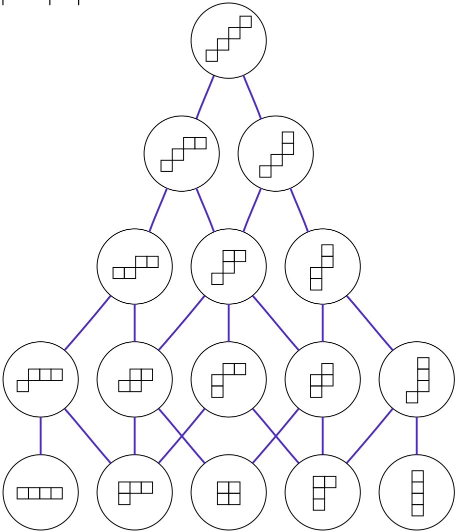
Definition. Let A, B be skew shapes. We say that

$$A \geq_s B \quad \text{if} \quad s_A - s_B \quad \text{is Schur-positive.}$$

Original goal: characterize the Schur-positivity order $\geq_s$ in terms of skew shapes.

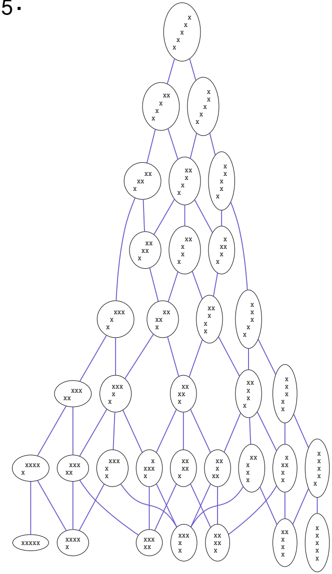
Example of a Schur-positivity poset

If $B \leq_s A$ then $|A| = |B|$.
Call the resulting ordered set P_n.
Then P_4:

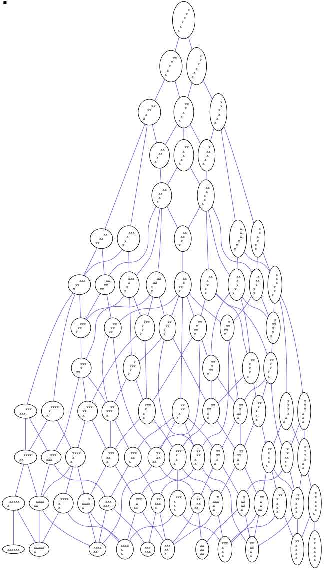


More examples

P_5:



P_6:



Known properties: Sufficient conditions

Sufficient conditions for $A \geq_s B$:

- Alain Lascoux, Bernard Leclerc, Jean-Yves Thibon (1997)
- Andrei Okounkov (1997)
- Sergey Fomin, William Fulton, Chi-Kwong Li, Yiu-Tung Poon (2003)
- Anatol N. Kirillov (2004)
- Thomas Lam, Alex Postnikov, Pavlo Pylyavskyy (2005)
- François Bergeron, Riccardo Biagioli, Mercedes Rosas (2006)
- McN., Steph van Willigenburg (2009, 2012)
- ...

Necessary conditions for Schur-positivity

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Notation. Write $\lambda \preccurlyeq \mu$ if λ is less than or equal to μ in dominance order, i.e.

$$\lambda_1 + \cdots \lambda_i \leq \mu_1 + \cdots \mu_i \text{ for all } i.$$

Examples. $331 \prec 421$ $\quad 21 \prec 32$ $\quad 33 \not\preccurlyeq 411$

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Examples. $331 \prec 421 \qquad 21 \prec 32 \qquad 33 \not\preccurlyeq 411$

Theorem [McN. (2008)]. Let A and B be skew shapes. If $s_A - s_B$ is Schur-positive, then

$$\text{rows}_k(A) \preccurlyeq \text{rows}_k(B) \text{ for all } k.$$

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Examples. $331 \prec 421$ $21 \prec 32$ $33 \not\preccurlyeq 411$

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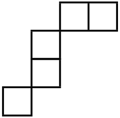
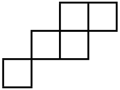
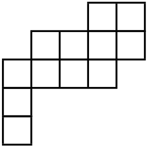
$A =$  $B =$ 

$$\begin{aligned} s_A &= s_{41} + s_{32} + 2s_{311} + s_{221} + s_{2111} \\ s_B &= s_{41} + 2s_{32} + s_{311} + s_{221} \end{aligned}$$

So $s_A - s_B$ is not Schur-positive but $\text{supp}_s(A) \supseteq \text{supp}_s(B)$.

Equivalent to row overlap conditions

Let $\text{rects}_{k,\ell}(A)$ denote the number of $k \times \ell$ rectangular subdiagrams contained inside A.

$A =$  $\text{rects}_{3,1}(A) = 2$, $\text{rects}_{2,2}(A) = 3$, etc.

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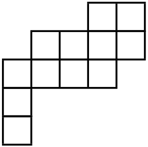
$A =$  $\mathrm{rects}_{3,1}(A) = 2$, $\mathrm{rects}_{2,2}(A) = 3$, etc.

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$$\boxed{s_A - s_B \text{ is Schur-pos.}} \quad \Rightarrow \quad \boxed{\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)} \quad \Rightarrow \quad \boxed{\begin{array}{l} \mathrm{rows}_k(A) \preccurlyeq \mathrm{rows}_k(B)\ \forall k \\ \mathrm{cols}_\ell(A) \preccurlyeq \mathrm{cols}_\ell(B)\ \forall \ell \\ \mathrm{rects}_{k,\ell}(A) \leq \mathrm{rects}_{k,\ell}(B)\ \forall k, \ell \end{array}}$$

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MIND THE GAP

Converse is very false.

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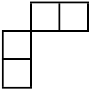
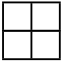
$A =$  $B =$ 

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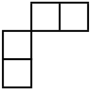
$A =$ 

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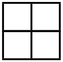
$B =$ 

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$s_B = s_{22}$

New Goal: Find weaker algebraic conditions on A and B that imply the overlap conditions.

What algebraic conditions are being encapsulated by the overlap conditions?

Insight from a more general setting

Example. $\sum_{i<j<k} x_i^6 x_j^4 x_k^9$ is not symmetric but it is *quasisymmetric*. e.g.

$$\text{coeff. of } x_1^6 x_2^4 x_3^9 = \text{coeff. of } x_5^6 x_9^4 x_{2014}^9.$$

Definition. A formal power series f in variables $x_1, x_2, \ldots$ is quasisymmetric if for all

- sequences $a_1, a_2, \ldots, a_k$ of exponents, and
- sequences $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$ of indices,

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Bases.

- Monomial quasisymmetric functions M_α:
  Given a *composition* $\alpha = (\alpha_1, \ldots, \alpha_k)$, e.g. $\alpha = (6, 4, 9)$,

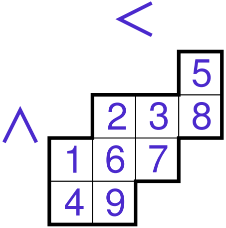
$$M_\lambda = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k}.$$

- Gessel's fundamental quasisymmetric functions F_α, e.g,

$$F_{32} = M_{32} + M_{212} + M_{122} + M_{1112} + M_{311} + M_{2111} + M_{1211} + M_{11111}.$$

F-basis of quasisymmetric functions

- Skew shape A.
- Standard Young tableau (SYT) T of shape A.



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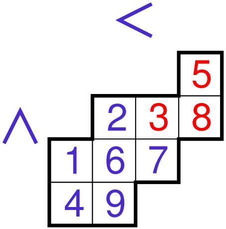

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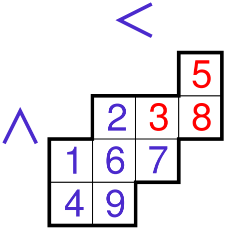

Then s_A expands in the basis of fundamental quasisymmetric functions as

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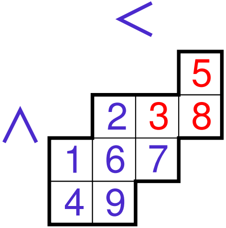

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Facts.

- The F form a basis for the quasisymmetric functions.
- So notions of F-positivity and F-support make sense.
- Schur-positivity implies F-positivity (converse fails at $n = 4$).
- $\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)$ implies $\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B)$

New results: filling the gap

Theorem. [McN. (2013)]

$$\boxed{s_A - s_B \text{ is Schur-pos.}} \Rightarrow \boxed{\mathrm{supp}_s(A) \supseteq \mathrm{supp}_s(B)}$$

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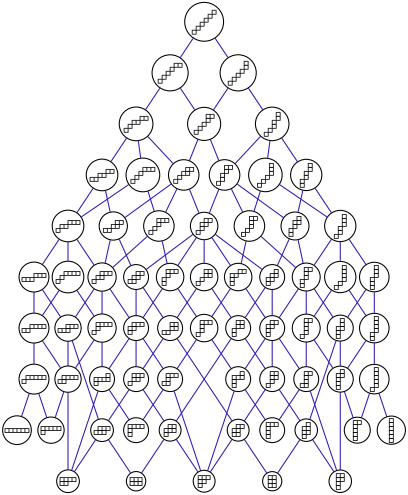
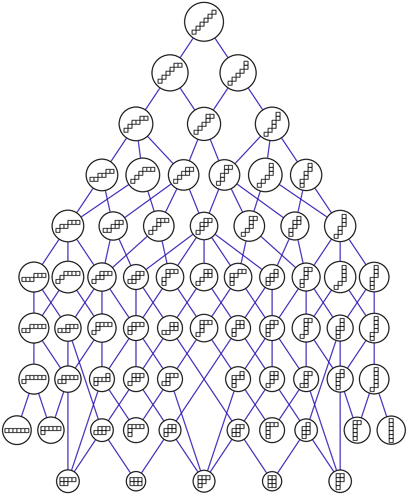
Evidence. Conjecture is true for:

- $n \leq 12$ (others fail already at $n = 4$);
- F-multiplicity-free skew shapes (as determined by Christine Bessenrodt and Steph van Willigenburg (2013));
- horizontal strips; ribbons whose rows all have length at least 2.

$n = 6$ example



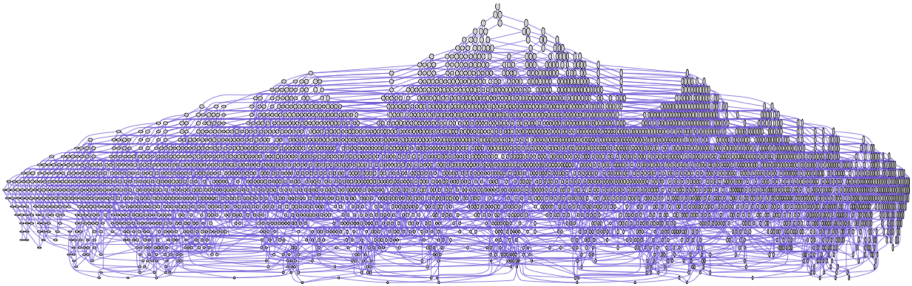
F-support containment



Row overlap reverse dominance

$n = 12$

$n = 12$ case has 12,042 edges.



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