# Exam 4 Solutions

1. (a)

| Step | $x$ | approximate $y$-value | $\Delta y = \text{slope} \cdot \Delta x$ ($y' = y + xy$) |
|---|---|---|---|
| 0 | 0 | 1 | $(1+0\cdot 1)(.1) = .1$ |
| 1 | 0.1 | $1+.1 = 1.1$ | $(1.1+(0.1)(1.1))(.1) = .121$ |
| 2 | 0.2 | $1.1+.121 = 1.221$ | $(1.221+(0.2)(1.221))(.1) = .14652$ |
| 3 | 0.3 | $1.221+.14652 \approx 1.368$ | $(1.368+(0.3)(1.368))(.1) \approx .178$ |
| 4 | 0.4 | $1.368+.178 = 1.546$ | $(1.546+(0.4)(1.546))(.1) \approx .216$ |
| 5 | 0.5 | $1.546+.216 = 1.762$ | |

$y(0.5) \approx 1.762$

(b) The slope field shows that the solution curves are concave up in the first quadrant, so a soln. curve through $(0,1)$ will be concave up to the right of the $y$-axis. Thus the tangent lines will lie beneath the soln., and our estimate is an underestimate.

(c) We have

$$\frac{dy}{dx} = y + xy = y(1+x)$$

$$\int \frac{dy}{y} = \int (1+x)\,dx \quad (y \neq 0)$$

$$\ln|y| = x + \tfrac{1}{2}x^2 + C$$

$$|y| = e^{x+\frac{1}{2}x^2+C} = e^C e^{x+\frac{1}{2}x^2}$$

We check that $y=0$ is also a solution, so $y = Ae^{x+\frac{1}{2}x^2}$, $A$ any constant, is the general solution. Plugging in our initial condition gives

$$1 = Ae^0 \Rightarrow A = 1,$$

so the exact solution is $y = e^{x+\frac{1}{2}x^2}$. We have $y(0.5) = e^{.5+.5(.5)^2} \approx 1.868$. Thus the error in using our approx. in (a) is $\approx 1.868 - 1.762 = .106$.

2. (a) We know the Taylor series for $e^t$ about $t=0$ is $\sum_{n=0}^{\infty} \frac{t^n}{n!}$, so the Taylor series for $te^t$ about $t=0$ is

$$t \cdot \sum_{n=0}^{\infty} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!}.$$

(other answers possible if change index accordingly)

The general term is $\frac{t^{n+1}}{n!}$.

(b) We have

$$\int_0^x te^t\,dt = \int_0^x \left(t + \frac{t^2}{1!} + \frac{t^3}{2!} + \frac{t^4}{3!} + \dots\right) dt$$

$$= \left[\frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{2!\cdot 4} + \frac{t^5}{3!\cdot 5} + \dots\right]_0^x$$

$$= \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{2!\cdot 4} + \frac{x^5}{3!\cdot 5} + \dots = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!\,(n+2)}$$

(c) We note that the sum $\frac{1}{2}+\frac{1}{3}+\frac{1}{4(2!)}+\frac{1}{5(3!)}+\frac{1}{6(4!)}+\dots$ is precisely the Taylor series we found in (b) with $x=1$. So

$$\text{sum} = \int_0^1 te^t\,dt \underset{\substack{\uparrow\\ \text{use integration by parts}}}{=} \left[te^t - e^t\right|_0^1 = e - e - (0-1) = \boxed{1}.$$

3.(a) Let $f(x)=\sin x$. Then

| | |
|---|---|
| $f'(x)=\cos x$ | $f(-\frac{\pi}{3}) = \sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$ |
| $f''(x) = -\sin x$ | $f'(-\frac{\pi}{3}) = \cos(-\frac{\pi}{3}) = \frac{1}{2}$ |
| $f'''(x) = -\cos x$ | $f''(-\frac{\pi}{3}) = -\sin(-\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ |
| | $f'''(-\frac{\pi}{3}) = -\cos(-\frac{\pi}{3}) = -\frac{1}{2}$ |

and the Taylor polynomial of degree 3 approx. $\sin x$ for $x$ near $a=-\frac{\pi}{3}$ is

$$P_3(x) = -\frac{\sqrt{3}}{2} + \frac{1}{2}\left(x+\frac{\pi}{3}\right) + \frac{\sqrt{3}}{2(2!)}\left(x+\frac{\pi}{3}\right)^2 - \frac{1}{2(3!)}\left(x+\frac{\pi}{3}\right)^3.$$

(b) We have $f^{(4)}(x)=\sin x$, and this fcn. is maximized on $[-\frac{2\pi}{3}, -\frac{\pi}{3}]$ at $x=-\frac{\pi}{2}$, where it has value $\sin(-\frac{\pi}{2}) = -1$. Thus $M=1$. The Lagrange error bound is

$$\text{Error} \le \frac{1}{4!}\left|-\frac{2\pi}{3}+\frac{\pi}{3}\right|^4 = \frac{1}{4!}\cdot\left(\frac{\pi}{3}\right)^4.$$

4. (a) Suppose that a straight line is a soln. to the diff. eqn. $\frac{dy}{dx} = x - \frac{1}{2}y$. Since the general eqn. for a straight line is $y=ax+b$, we can plug the appropriate pieces into our diff. eqn. and solve for $a$ and $b$:

$$y = ax+b \Rightarrow \frac{dy}{dx} = a$$

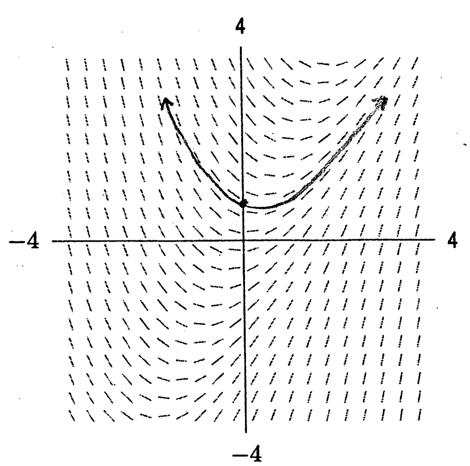
$$\frac{dy}{dx} = x - \frac{1}{2}y \Rightarrow a = x - \frac{1}{2}(ax+b) \Rightarrow a = x\left(1-\frac{a}{2}\right) - \frac{b}{2}$$

$$1-\frac{a}{2} = 0 \quad \text{and} \quad a = -\frac{b}{2}$$

$$a = 2 \longrightarrow b = -4$$

So if $y=ax+b$ is to be a soln. to our diff. eqn., it must be the line $y=2x-4$.

(b)




(c) many possible soln. curves; one is sketched on slope field