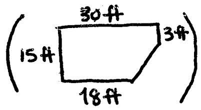
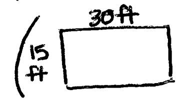
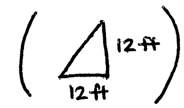
Exam 3 Solutions

1. (a) We slice parallel to the bottom of the pool. Note that for such a cross-section, we have

area 

= area 

− area 

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So the volume of a slab is

h is distance from top of pool

$$\text{vol(slab)} \approx \left(30\cdot 15 - \tfrac{1}{2}\cdot 12\cdot 12\right)\cdot \Delta h \text{ ft}^3 = (450-72)\,\Delta h \text{ ft}^3 = 378\,\Delta h \text{ ft}^3$$

and

$$\text{weight(slab)} \approx 378\,\Delta h \cdot 62.4 \text{ lbs.}$$

The work to move a slab a distance of h feet is approximately $(378)(62.4)h\Delta h$ ft-lbs. The work to pump all the water out is approximately $\Sigma\,(378)(62.4)h\Delta h$ ft-lbs, and letting $\Delta h \to 0$ gives

$$\begin{aligned}\text{Total work} &= \int_0^{10} (378)(62.4)\,h\,dh \\ &= (378)(62.4)\left[\frac{h^2}{2}\right|_0^{10} \\ &= (378)(62.4)(50) \approx \boxed{1{,}179{,}360 \text{ ft-lbs.}}\end{aligned}$$

(b) The pressure is the same at every point on the bottom of the pool, so

$$\text{Pressure} = (62.4 \text{ lb/ft}^3)(10 \text{ ft}) = 624 \text{ lb/ft}^2$$

and

$$\text{Force} = (624 \text{ lb/ft}^2)(378 \text{ ft}^2) = \boxed{235{,}872 \text{ lbs}}.$$

2. To find the radius of convergence, we use the Ratio Test:

$$\frac{\left|\dfrac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}\right|}{\left|\dfrac{(-3)^n x^n}{\sqrt{n+1}}\right|} = \frac{3^{n+1}|x|^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{3^n|x|^n} = 3|x|\sqrt{\frac{n+1}{n+2}}$$

$$\lim_{n\to\infty} 3|x|\sqrt{\frac{n+1}{n+2}} = 3|x|\lim_{n\to\infty}\sqrt{\frac{n+1}{n+2}}\cdot\sqrt{\frac{1/n}{1/n}} = 3|x|\lim_{n\to\infty}\sqrt{\frac{1+1/n}{1+2/n}} = 3|x|$$

The series converges when $3|x|<1$, or $|x|<\frac{1}{3}$. The radius of convergence is $\boxed{R=\frac{1}{3}}$.

Now we need to check the endpoints:

$x=-\frac{1}{3}$: $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n\left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{\left(-3\cdot-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{1^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$

Easy way: $\displaystyle\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, a p-series with $p=\frac{1}{2}<1$, so diverges

Another way: Use LCT with $b_n=\frac{1}{\sqrt{n}}$, $a_n=\frac{1}{\sqrt{n+1}}$. Note that $a_n>0$ and $b_n>0$.

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}}\cdot\sqrt{\frac{1/n}{1/n}} = \lim_{n\to\infty}\sqrt{\frac{1}{1+1/n}} = 1$$

By the LCT, the two series have the same behavior. Since $\displaystyle\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$ is a p-series with $p=\frac{1}{2}<1$, it diverges; hence $\displaystyle\sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$ diverges.

Note: The Comparison Test, using $\frac{1}{\sqrt{n}}$, does <u>not</u> work. We have $n+1>n \Rightarrow \sqrt{n+1}>\sqrt{n} \Rightarrow \frac{1}{\sqrt{n+1}}<\frac{1}{\sqrt{n}}$, and this gives no information since $\Sigma\frac{1}{\sqrt{n}}$ diverges.

$x=\frac{1}{3}$: $\sum_{n=0}^{\infty} \frac{(-3)^n(\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3\cdot\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

in "Note"

This is an alternating series, with $a_n = \frac{1}{\sqrt{n+1}}$. We saw on the front that the terms are decreasing, and we have $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{\sqrt{1+1/n}} = 0$, so the series converges by the Alternating Series Test.

The interval of convergence is $(-\frac{1}{3}, \frac{1}{3}]$.

3. (a) True (b) False (c) False (need $k \neq 0$)

4. Consider the series $\sum_{n=1}^{\infty} \frac{1}{5^n}$. This is a geometric series with first term $\frac{1}{5}$ ($a=\frac{1}{5}$) and common ratio $\frac{1}{5}$ ($x=\frac{1}{5}$). Thus the sum of the series is $\frac{1/5}{1-1/5} = \frac{1/5}{4/5} = \frac{1}{4}$.

Next consider the series $\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. We have two options (at least!):

- Wonder why the denominator is written in this form, guess that it's telescoping, do partial fractions: $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Rightarrow 1 = A(n+1)+Bn \Rightarrow 1 = n(A+B)+A$
$\therefore A=1, B=-1$

So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) + \ldots$

The nth partial sum is $S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$, and $\lim_{n\to\infty} S_n = 1$. So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

- Write out some partial sums and look for a pattern.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots$$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

$S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$

$S_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{48}{60} = \frac{4}{5}$

$\vdots$

$S_n = \frac{n}{n+1}$ Since $\lim_{n\to\infty} S_n = 1$, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Thus both series converge, and we have

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{5^n}\right) = 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{5^n} = 3(1) + \frac{1}{4} = \boxed{\frac{13}{4}}.$$

5. (a) Note that $\sin k$ takes on various positive and negative values, so this is not an alternating series. We can check for absolute convergence using the Comparison Test:

$\left|\frac{\sin k}{k^3}\right| = \frac{|\sin k|}{k^3} \leq \frac{1}{k^3}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ is a p-series w/ $p=3>1$, it converges.

By the Comparison Test, $\sum_{k=1}^{\infty} \frac{|\sin k|}{k^3}$ converges. Absolute convergence implies convergence, so $\sum_{k=1}^{\infty} \frac{\sin k}{k^3}$ converges.

(b) We have $8n^2 > 8n^2 - 3n \Rightarrow \sqrt[3]{8n^2} > \sqrt[3]{8n^2-3n} \Rightarrow \frac{1}{\sqrt[3]{8n^2}} < \frac{1}{\sqrt[3]{8n^2-3n}}$. Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{8n^2}} = \sum_{n=1}^{\infty} \frac{1}{2n^{2/3}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ is a p-series w/ $p=\frac{2}{3}<1$ (so diverges), $\sum_{n=1}^{\infty} \frac{1}{2n^{2/3}}$ diverges. By the Compariso Test, the original series diverges. (can also do using LCT)

(c) We use the Integral Test with $f(x) = \frac{1}{(x+1)[\ln(x+1)]^2}$. We note that $f(x)$ is positive and continuous for $x \geq 1$ (at least). We find $f'(x) = -\frac{1}{[(x+1)(\ln(x+1))]^2} - \frac{2}{(x+1)^2(\ln(x+1))^3}$. Both denominators are positive for $x \geq 1$, so $f'(x) < 0$ for $x \geq 1$, i.e., $f(x)$ is decreasing for $x \geq 1$.

$$\int_1^{\infty} \frac{1}{(x+1)[\ln(x+1)]^2}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u^2}\,du = \lim_{b\to\infty} \int_{\ln 2}^{b} u^{-2}\,du = \lim_{b\to\infty} -\frac{1}{u}\Big|_{\ln 2}^{b} = \lim_{b\to\infty}\left(-\frac{1}{b} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

$u = \ln(x+1)$

$du = \frac{1}{x+1}\,dx$

By the Integral Test, $\sum_{k=1}^{\infty} \frac{1}{(k+1)[\ln(k+1)]^2}$ converges.