ELEC 320
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FREQUENCY DOMAIN AND FOURIER SERIES

- What is "Fourier analysis"?

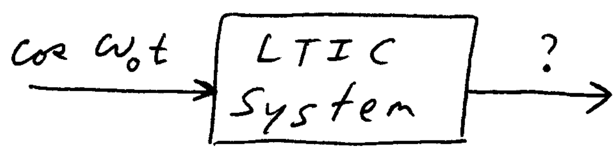
<u>Loose statement:</u> "Any" signal can be expressed as a sum of sine waves with different frequencies, amplitudes, and phases.

Fourier <u>series</u> is for periodic signals.

Fourier <u>transform</u> is for non-periodic signals.

- Why do we use <u>sine</u> waves?

Recall this property of sine waves when applied to linear, time-invariant, continuous-time (LTIC) systems:



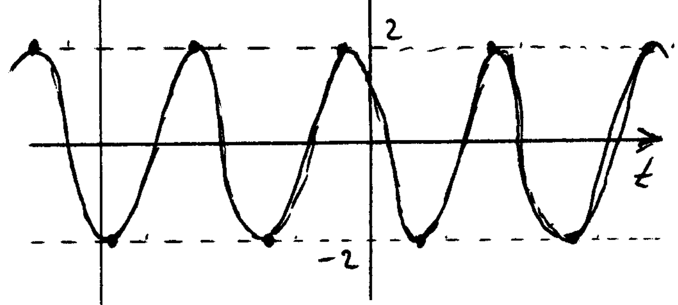
- Sinusoids are easy to deal with in LTIC systems.
- "Any" input signal is a <u>sum</u> of sine waves (by Fourier)

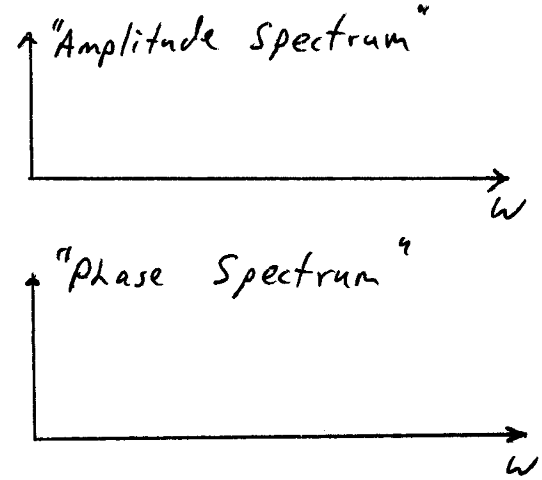
$\therefore$ Linearity $\Rightarrow$ System output is the sum of responses to individual sine waves.

(This leads to <u>frequency response</u> of systems!)

Time Domain vs. Frequency Domain

$x(t) = 2\cos(5t + 30°)$

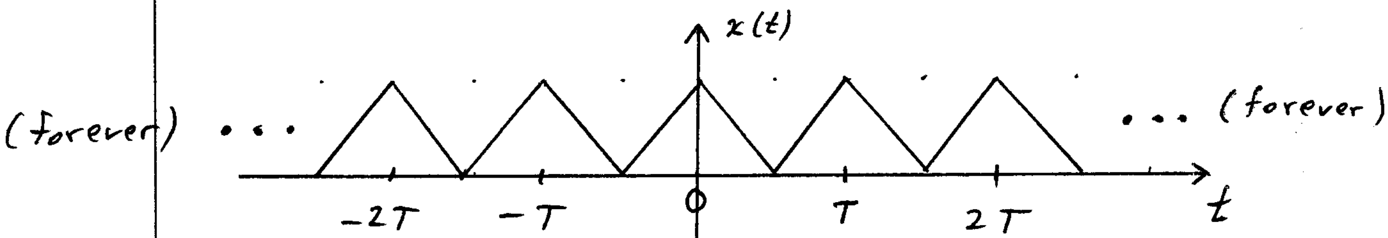




- Time and frequency domains are two ways to view the same signal.
- Fourier analysis lets us represent "any" signal in the frequency domain.

FOURIER SERIES

- What is a periodic signal? $x(t) = x(t+T)$



Which sine wave frequencies can we use to build this $x(t)$?

Fundamental frequency: $\omega_0 = \frac{2\pi}{T}$ (rad/sec)

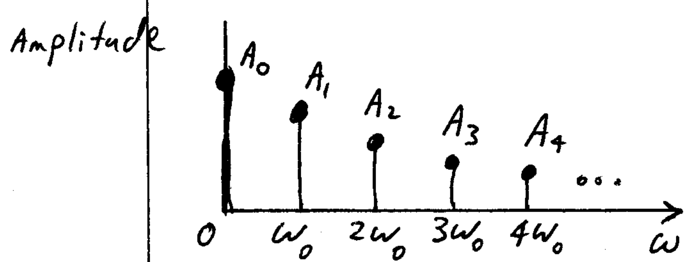
Harmonics: $2\omega_0,\ 3\omega_0,\ \ldots,\ n\omega_0,\ \ldots$

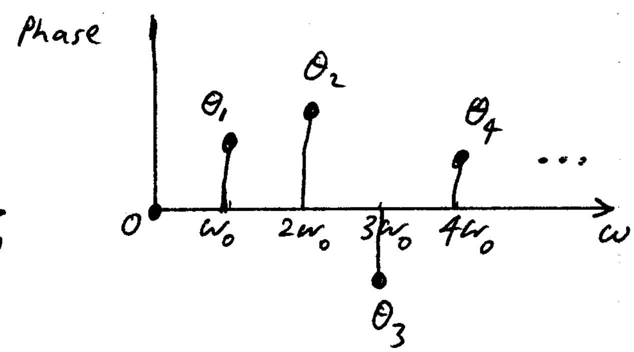
Trigonometric Fourier series:

$$x(t) = A_0 + A_1 \cos(\omega_0 t + \theta_1) + A_2 \cos(2\omega_0 t + \theta_2) + A_3 \cos(3\omega_0 t + \theta_3) + \cdots$$

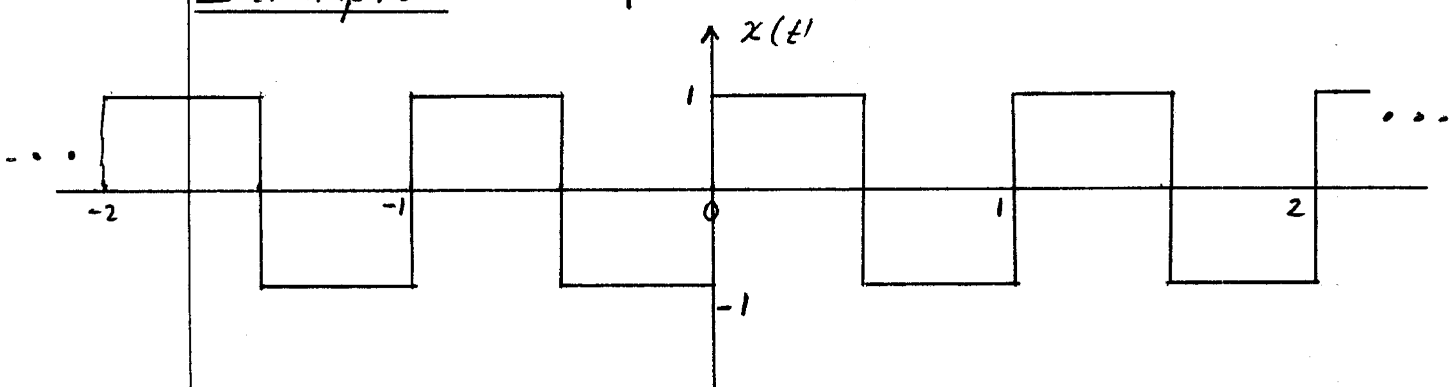
$$= A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n)$$

Frequency domain: (1-sided spectra)





Example: Square wave.



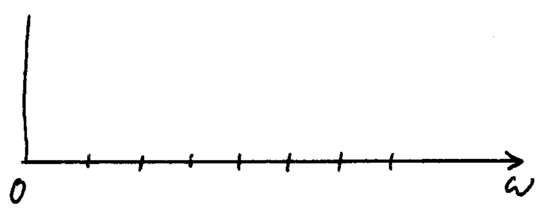
Period: $T =$ _____ Fundamental: $\omega_0 =$ _____

We found that:

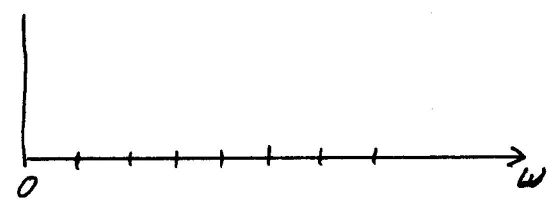
$$x(t) = \frac{4}{\pi}\sin(2\pi t) + \frac{4}{\pi}\cdot\frac{1}{3}\sin(2\pi \cdot 3t)$$
$$+ \frac{4}{\pi}\cdot\frac{1}{5}\sin(2\pi 5t) + \frac{4}{\pi}\cdot\frac{1}{7}\sin(2\pi 7t) + \cdots$$

Plot 1-sided amplitude and phase spectra for this square wave:

Amplitude



Phase



How can we find the Fourier series coefficients $\{A_n, \theta_n : n = 0, 1, 2, \ldots\}$ for an arbitrary, periodic signal?

It is simpler to use complex exponentials in the Fourier series rather than cosines.

- → Easier to compute
- → Better for EE applications

Recall Euler's formula & some derived relations:

$$e^{j\varphi} = \cos\varphi + j\sin\varphi \qquad \text{(Euler's formula)}$$

$$e^{-j\varphi} =$$

$$e^{j\varphi} + e^{-j\varphi} =$$

$$e^{j\varphi} - e^{-j\varphi} =$$

$$\cos\varphi =$$

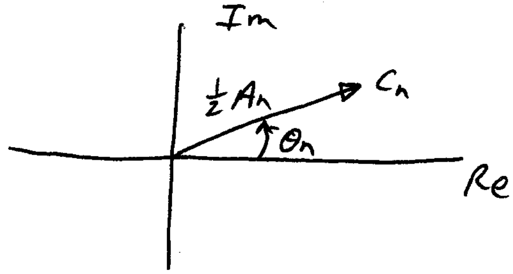
$$\sin\varphi =$$

Apply to terms of trig. Fourier series:

$$A_n \cos(n\omega_0 t + \theta_n) = \tfrac{1}{2} A_n e^{j(n\omega_0 t + \theta_n)} + \tfrac{1}{2} A_n e^{-j(n\omega_0 t + \theta_n)}$$

$$= \underbrace{\left(\tfrac{1}{2} A_n e^{j\theta_n}\right)}_{C_n} \underbrace{e^{jn\omega_0 t}}_{\substack{\text{Frequency} \\ +n\omega_0}} + \underbrace{\left(\tfrac{1}{2} A_n e^{-j\theta_n}\right)}_{C_{-n}} \underbrace{e^{-jn\omega_0 t}}_{\substack{\text{Frequency} \\ -n\omega_0}}$$

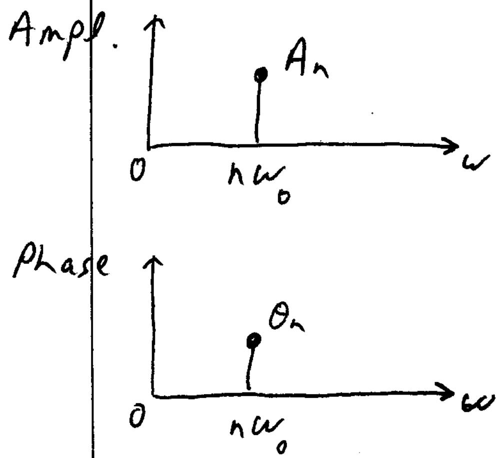
$$= C_n e^{jn\omega_0 t} + C_{-n} e^{-jn\omega_0 t}$$



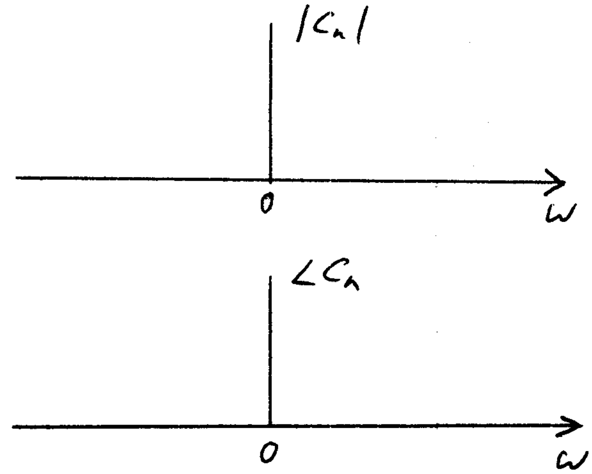
$$|C_n| = \tfrac{1}{2} A_n$$

$$\angle C_n = \theta_n$$

1- sided Spectrum



2- sided Spectrum



Need "positive" and "negative" frequency to produce a single real cosine:

$$\cos \omega t = \frac{1}{2} e^{+j\omega t} + \frac{1}{2} e^{-j\omega t}$$

$\Rightarrow$ Exponential Fourier series: $\left(\begin{array}{l} x(t) = x(t+T) \\ \omega_0 = \frac{2\pi}{T} \end{array} \right)$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \qquad -\infty < t < \infty$$

The coefficients c_n for a given signal $x(t)$ are obtained from

$$c_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} \, dt$$

Integrate over one complete period

Ex: $[0, T]$, $[-\frac{T}{2}, \frac{T}{2}]$, $[-\frac{T}{4}, \frac{3T}{4}]$, etc

Example: Find exponential Fourier series coefficients C_n for the square wave on page ③ of these notes.

$$T = 1, \quad \text{so} \quad \omega_0 = \frac{2\pi}{1} = 2\pi$$

$$C_n = \int_0^1 x(t)\, e^{-j2\pi n t}\, dt \quad \text{where} \quad x(t) = \begin{cases} 1, & 0 \le t \le .5 \\ -1, & .5 \le t \le 1 \end{cases}$$

$$= \int_0^{.5} (1)\cdot e^{-j2\pi n t}\, dt + \int_{.5}^{1} (-1)\, e^{-j2\pi n t}\, dt$$

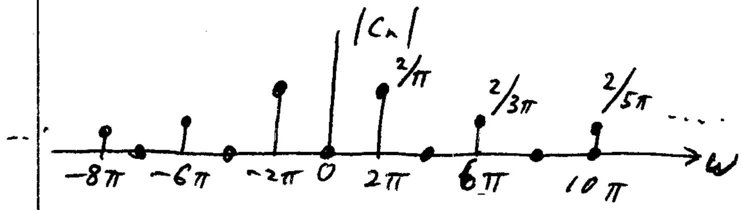
$$= \frac{1}{-j2\pi n} e^{-j2\pi n t}\Big|_0^{.5} + \frac{-1}{-j2\pi n} e^{-j2\pi n t}\Big|_{.5}^{1}$$

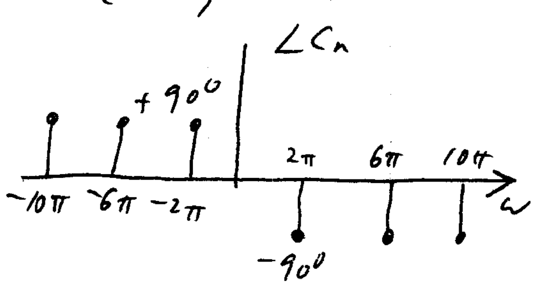
$$= \frac{1}{-j2\pi n}\left[e^{-j\pi n} - 1\right] + \frac{1}{j2\pi n}\left[e^{-j2\pi n} - e^{-j\pi n}\right]$$

Note that $e^{-j\pi n} = \begin{cases} -1 & \text{for } n \text{ odd} \\ +1 & \text{for } n \text{ even} \end{cases}$

and $e^{-j2\pi n} = +1$ for all integer n.

$$\therefore \quad C_n = \begin{cases} \dfrac{2}{j\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} = \begin{cases} \dfrac{2}{\pi|n|} \angle \pm 90^\circ, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$





[See MATLAB program fstrun.m on Web page.]