RC Circuit Steady-State Analysis in Time-Domain

Series RC circuit, $v_{out}(t)$ measured across C,
sinusoidal voltage source $v_{in}(t)$: $v_{in}(t) = A_i \cos(\omega t)$ $v_{out}(t) = A_o \cos(\omega t + \vartheta)$

KCL:
$$\frac{v_{out}(t) - v_{in}(t)}{R} + C\frac{d\,v_{out}}{d\,t} = 0$$
$$v_{out}(t) - v_{in}(t) + RC\frac{d\,v_{out}}{d\,t} = 0$$
$$A_o \cos(\omega t + \vartheta) - A_i \cos(\omega t) + RC[-A_o\omega \sin(\omega t + \vartheta)] = 0$$

Trig. identities: $\cos(x+y) = \cos x \cos y - \sin x \sin y, \quad \sin(x+y) = \sin x \cos y + \cos x \sin y$

Euler's equations:
$$e^{jx} = \cos x + j\sin x, \quad e^{-jx} = \cos x - j\sin x \quad [\text{Special case}: e^{j\pi} = \text{-}1]$$
$$\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right) = \mathrm{Re}\left(e^{jx}\right), \quad \sin x = \frac{1}{j2}\left(e^{jx} - e^{-jx}\right) = \mathrm{Im}\left(e^{jx}\right) = \mathrm{Re}\left(-je^{jx}\right)$$

Relate to trig. identities:
$$e^{j(x+y)} = \cos(x+y) + j\sin(x+y) = e^{jx}e^{jy} = (\cos x + j\sin x)(\cos y + j\sin y)$$
$$= [\cos x \cos y - \sin x \sin y] + j[\sin x \cos y + \cos x \sin y]$$

Apply trig. identities to KCL equation:

$$A_o[\cos(\vartheta)\cos(\omega t) - \sin(\vartheta)\sin(\omega t)] - A_i\cos(\omega t) - RCA_o\omega[\sin(\vartheta)\cos(\omega t) + \cos(\vartheta)\sin(\omega t)] = 0$$
$$\{A_o[\cos(\vartheta) - RC\omega\sin(\vartheta)] - A_i\}\cos(\omega t) - \{A_o[\sin(\vartheta)] + RCA_o\omega[\cos(\vartheta)]\}\sin(\omega t) = 0$$

In order for this to be true for <u>all</u> time values, t:
$$A_o[\cos(\vartheta) - RC\omega\sin(\vartheta)] - A_i = 0$$
$$A_o[\sin(\vartheta)] + RCA_o\omega[\cos(\vartheta)] = 0$$

Apply Euler's equations to KCL equation:

$$A_o\cos(\omega t + \vartheta) - A_i\cos(\omega t) + RC[-A_o\omega\sin(\omega t + \vartheta)] = 0$$
$$\mathrm{Re}\{A_o\exp[j(\omega t + \vartheta)] - A_i\exp[j\omega t] + RC\; jA_o\omega\exp[j(\omega t + \vartheta)]\} = 0$$
$$\mathrm{Re}\left\{\left[\left(A_oe^{j\vartheta}\right) - A_i + \frac{R}{\left(1/j\omega C\right)}\left(A_oe^{j\vartheta}\right)\right]e^{j\omega t}\right\} = 0$$
$$\mathrm{Re}\left\{\left[\frac{\left(A_oe^{j\vartheta}\right) - A_i}{R} + \frac{\left(A_oe^{j\vartheta}\right)}{\left(1/j\omega C\right)}\right]e^{j\omega t}\right\} = 0 \quad \text{The following must be true for all } t:$$
$$\mathrm{Re}\left\{\left[\frac{\left(A_oe^{j\vartheta}\right) - A_i}{R} + \frac{\left(A_oe^{j\vartheta}\right)}{\left(1/j\omega C\right)}\right]\right\}\cos(\omega t) - \mathrm{Im}\left\{\left[\frac{\left(A_oe^{j\vartheta}\right) - A_i}{R} + \frac{\left(A_oe^{j\vartheta}\right)}{\left(1/j\omega C\right)}\right]\right\}\sin(\omega t) = 0$$