Natural and Step Responses for RLC Circuits

The natural and step responses of RLC circuits are described by second-order, linear differential equations with constant coefficients and constant "input" (or forcing function),

$$a \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + cx(t) = D,$$

(1)

where $a$, $b$, $c$, and $D$ are constants, and the initial values $x(0^+)$ and $\frac{dx(0^+)}{dt}$ are known (these are found by circuit analysis). We assume $a > 0$, without loss of generality.

As you know from MATH 212, the general solution for $t > 0$ is

$$x(t) = x_c(t) + x_p(t),$$

(2)

where $x_c(t)$ is the complementary solution to the homogeneous equation and $x_p(t)$ is a particular solution. The particular solution is

$$x_p(t) = \begin{cases} \frac{D}{c}, & c \neq 0 \\ \frac{D}{b} \frac{dt}{t}, & c = 0, b \neq 0 \\ \frac{D}{2a} t^2, & c = b = 0 \end{cases}.$$  

(3)

The homogeneous differential equation is

$$a \frac{d^2 x_c}{dt^2} + b \frac{dx_c}{dt} + cx_c(t) = 0.$$  

(4)

Its solution is determined by the characteristic (or auxiliary) equation,

$$as^2 + bs + c = 0,$$

(5)

which has roots

$$s_1 = -\frac{b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2},$$

(6)

$$s_2 = -\frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2}.$$  

(7)

The neper frequency is $\alpha = b/(2a)$ and the resonant frequency is $\omega_0 = \sqrt{c/a}$ (in radians/sec). If $c < 0$, then $\omega_0$ is imaginary and $s_1 > 0$ in (6).
There are three cases.

1. Real and unequal roots (overdamped): $\alpha^2 > \omega_0^2$

   $$x_c(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$  \hspace{1cm} (8)

   Both terms are decaying exponentials if $c > 0$.

2. Real and equal roots (critically damped): $\alpha^2 = \omega_0^2$

   $$x_c(t) = c_1 e^{s_1 t} + c_2 t e^{s_1 t}, \text{ where } s_1 = -\alpha = -\frac{b}{2a}$$  \hspace{1cm} (9)

3. Complex roots (underdamped): $\alpha^2 < \omega_0^2$

   The roots have the form

   $$s_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$$  \hspace{1cm} (10)

   where $\omega_d$ is the damped frequency (in radians/sec). Then

   $$x_c(t) = c_1 e^{-\alpha t} \cos(\omega_d t) + c_2 e^{-\alpha t} \sin(\omega_d t)$$  \hspace{1cm} (11)

   In the case that $\alpha = 0$, then $\omega_d = \omega_0$ and the solution is pure, undamped oscillation,

   $$x_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$  \hspace{1cm} (12)

   This is why $\omega_0$ is called the resonant frequency.

Summary:

- The general solution is in equation (2): $x(t) = x_c(t) + x_p(t)$.
- The complementary solution $x_c(t)$ is either (8), (9), or (11).
- The particular solution $x_p(t)$ is in (3). For the natural response, $D = 0$ and $x_p(t) = 0$.
- The constants $c_1$ and $c_2$ are determined from the initial conditions $x(0^+)$ and $\frac{dx(0^+)}{dt}$.
- Hints for finding initial conditions in RLC circuits:
  - Parallel RLC: Analyze voltages, so $x(t) = v(t)$. Then $v(0^+)$ is determined from the initial capacitor voltage, and $\frac{dv(0^+)}{dt}$ is determined from KCL and the initial inductor current.
  - Series RLC: Analyze currents, so $x(t) = i(t)$. Then $i(0^+)$ is determined from the initial inductor current, and $\frac{di(0^+)}{dt}$ is determined from KVL and the initial capacitor voltage.