ELEC 226
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Natural and Step Responses for RLC Circuits

The natural and step responses of RLC circuits are described by second-order, linear differential equations with constant coefficients and constant "input" (or forcing function),

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c\,x(t) = D, \tag{1}$$

where a, b, c, and D are constants, and the initial values $x(0^+)$ and $\frac{dx(0^+)}{dt}$ are known (these are found by circuit analysis). We assume $a > 0$, without loss of generality.

As you know from MATH 212, the general solution for $t > 0$ is

$$x(t) = x_c(t) + x_p(t), \tag{2}$$

where $x_c(t)$ is the complementary solution to the homogeneous equation and $x_p(t)$ is a particular solution. The particular solution is

$$x_p(t) = \begin{cases} \frac{D}{c}, & c \neq 0 \\ \frac{D}{b}t, & c = 0, b \neq 0 \\ \frac{D}{2a}t^2, & c = b = 0 \end{cases} . \tag{3}$$

The homogeneous differential equation is

$$a\frac{d^2x_c}{dt^2} + b\frac{dx_c}{dt} + c\,x_c(t) = 0. \tag{4}$$

Its solution is determined by the characteristic (or auxiliary) equation,

$$as^2 + bs + c = 0, \tag{5}$$

which has roots

$$s_1 = -\frac{b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \tag{6}$$

$$s_2 = -\frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2}. \tag{7}$$

The neper frequency is $\alpha = b/(2a)$ and the resonant frequency is $\omega_0 = \sqrt{c/a}$ (in radians/sec). If $c < 0$, then ω_0 is imaginary and $s_1 > 0$ in (6).

There are three cases.

1. Real and unequal roots (overdamped): $\alpha^2 > \omega_0^2$

$$x_c(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} \tag{8}$$

Both terms are *decaying* exponentials if $c > 0$.

2. Real and equal roots (critically damped): $\alpha^2 = \omega_0^2$

$$x_c(t) = c_1 e^{s_1 t} + c_2\, t\, e^{s_1 t}, \quad \text{where} \;\; s_1 = -\alpha = -\frac{b}{2a} \tag{9}$$

3. Complex roots (underdamped): $\alpha^2 < \omega_0^2$

The roots have the form

$$s_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\,\omega_d \tag{10}$$

where ω_d is the damped frequency (in radians/sec). Then

$$x_c(t) = c_1 e^{-\alpha t}\cos(\omega_d t) \; + \; c_2 e^{-\alpha t}\sin(\omega_d t) \tag{11}$$

In the case that $\alpha = 0$, then $\omega_d = \omega_0$ and the solution is pure, undamped oscillation,

$$x_c(t) = c_1 \cos(\omega_0 t) \; + \; c_2 \sin(\omega_0 t). \tag{12}$$

This is why ω_0 is called the *resonant frequency.*

Summary:

- The general solution is in equation (2): $x(t) = x_c(t) + x_p(t)$.
- The complementary solution $x_c(t)$ is either (8), (9), or (11).
- The particular solution $x_p(t)$ is in (3). For the natural response, $D = 0$ and $x_p(t) = 0$.
- The constants c_1 and c_2 are determined from the initial conditions $x(0^+)$ and $\frac{dx(0^+)}{dt}$.
- Hints for finding initial conditions in RLC circuits:
  - Parallel RLC: Analyze voltages, so $x(t) = v(t)$. Then $v(0^+)$ is determined from the initial capacitor voltage, and $\frac{dv(0^+)}{dt}$ is determined from KCL and the initial inductor current.
  - Series RLC: Analyze currents, so $x(t) = i(t)$. Then $i(0^+)$ is determined from the initial inductor current, and $\frac{di(0^+)}{dt}$ is determined from KVL and the initial capacitor voltage.